An application of Rantzer's Dual Lyapunov Theorem to Decentralized Navigation

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***Abstract*— We provide a connection between Rantzer's dual Lyapunov Theorem that appeared in [18] with Decentralized Navigation Functions (DNFs). It is shown that when the agents' control law does not contain an element that forces them to cooperate with the rest of the team once they have reached their desired goal, global convergence cannot be guaranteed. A sufficient condition for this to happen is derived based on Rantzer's Theorem. In particular, it is shown that agents are driven towards their goals provided that collisions between the team members tend to occur whenever agents are found sufficiently far from their desired destinations. This is derived based on the properties of the critical points of the DNF's imposed by Rantzer's Theorem. The result can be used as a new approach to guaranteed local-minima free decentralized control approaches.**

I. Introduction

The emerging use of large-scale multi-robot and multi-vehicle systems in various modern applications has raised recently the need for the design of control laws that force a team of multiple vehicles/robots (from now on called "agents") to achieve various goals. As the number of agents increases, centralized control designs fail to guarantee robustness and are harder to implement than decentralized approaches, which also provide a reduce in the computational complexity of the overall feedback scheme.

A closed loop approach for single robot navigation was proposed by Koditschek and Rimon [10], [20] in their seminal work. This navigation functions' framework handled single, point-sized, robot navigation. In [13],[14] this method was successfully extended to take into account the volume of each robot in a centralized multi-agent scheme, while a decentralized version of this work has been presented by the authors in [4] for multiple holonomic agents with global sensing capabilities and in [3] for the case of limited sensing capabilities. While in these papers the objective of the multi-agent system was convergence to non-cooperative equilibria with collision avoidance, convergence to cooperative equilibria (aka formation control) using decentralized navigation functions was dealt with in [5] for the case of sphere world agents, while point world-agents were taken into account in [2]. Decentralized navigation functions were also used for multiple UAV guidance in [1]. Moreover, numerous relative results on decentralized control of multi-agent kinematic systems have appeared recently in literature including formation [11],[16],[12] and consensus control schemes [19],[17].

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The stability analysis of the decentralized scheme in [4],[3] involved tools from classical Lyapunov theory and Morse theory. We used a machinery which allowed agents that had already reached their desired destination to cooperate with the rest of the team in the case of a possible collision. In this paper, we use a construction similar to the initial navigation function construction in [10]. Hence each agent no longer participates in the collision avoidance procedure if its initial condition coincides with its desired destination. As a result, the closed loop system can converge to critical points which are no longer guaranteed not to coincide with local minima. What we can hope for is that the agents converge to a certain subset of the state space containing the target locations. In this paper we provide sufficient conditions for this using Rantzer's dual Lyapunov Theorem [18]. We should note that the results of this paper are extended to the case of formation control in [6].

In [18], A. Rantzer presented a new convergence criterion for nonlinear systems, which involved the divergence of the vector field with respect to a certain positive function (called density function in [18]) instead of the time derivative of a positive definite function, as in the classical Lyapunov approach. Density functions can be considered as the dual of the classical Lyapunov functions, while the condition that the divergence is positive for almost all initial conditions as the dual of the requirement of the negative definiteness of the Lyapunov time derivative. The main advantage of this approach is the fact that convergence can be checked and proved for systems which are not asymptotically stable. The weaker notion of convergence introduced in [18] is used in this paper to derive a sufficient condition for navigation of the closed loop system to a subset of the workspace containing the target locations. The main motivation however of this paper, is to provide the first result connecting Rantzer's dual Lyapunov theory with the general problem of local minima avoidance in decentralized control. This can serve as a guideline for future research directions in decentralized control, such as collision free swarm aggregation where the existence of local minima is a major disadvantage [9],[8].

The rest of the paper is organized as follows: section II describes the system and the problem in hand. In section III the theory of [18] is reviewed and we proceed by presenting the Decentralized Navigation Functions framework used in this paper. In section IV, the convergence of the feedback control scheme is analyzed using Rantzer's Theorem, while section V includes computer simulations that support the derived results. The last section summarizes the conclusions of this paper and indicates further research directions.

II. System and Problem Statement

Consider a system of N agents operating in the same planar workspace $W \subset \mathbb{R}^2$. Let $q_i \in \mathbb{R}^2$ denote the position of agent i. The configuration space is spanned by $q = [q_1, \ldots, q_N]^T$. The motion of each agent is described by the single integrator:

$$\dot{q}_i = u_i, i \in \mathcal{N} = [1, \ldots, N] \tag{1}$$

where u_i denotes the velocity (control input) for each agent. The desired destinations of the agents are respectively denoted by the index d: $q_d = [q_{d1}, \ldots, q_{dN}]^T$. We consider cyclic agents of specific radius $r \geq 0$, which is common for each agent. The results can trivially be extended to the case of agents with not necessarily common radii. For $r = 0$, the problem in reduced to the degenerate case of point agents.

The objective of each agent is navigation from an initial position to a desired destination avoiding at the same time collisions with the other agents. Collision avoidance is meant in the sense that no intersections occur between the agents' discs. Thus we want to assure that

$$\|q_i(t) - q_j(t)\| > 2r, \forall i, j \in \mathcal{N}, i \neq j \tag{2}$$

for each time instant t.

Furthermore, we assume that each agent has only knowledge of the position of agents located in a cyclic neighborhood of specific radius d at each time instant, where $d > 2r$. This set $T_i = \{q : \|q - q_i\| \leq d\}$ is called the *sensing zone* of agent i. The control design is hence of the form

$$u_i = u_i\left(q_i, \{q_j, j \in S_i\}\right)$$

where $S_i = \{j \in \mathcal{N}, j \neq i : \|q_i - q_j\| \leq d\}$ the set of indices of agents that are located in the sensing zone of i at each time instant. Finally, the agents evolve in a spherical bounded planar workspace$W \triangleq \{q | \, \|q\| \leq R_W\} \subset \mathbb{R}^{2N}$, where R_W is the workspace radius. We assume that all agents have knowledge of the workspace boundary.

A possible conflict scenario is shown in Figure 1.

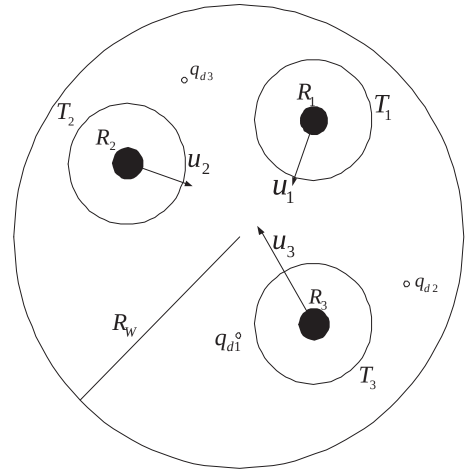

Fig. 1. A conflict scenario with three agents. Each agent i occupies a disc R_i(black discs) of radius r centered at q_i. The agents evolve in a bounded workspace of radius R_W. Each agent's sensing zone T_i(white discs) is centered at q_i and has radius d.

III. Mathematical Preliminaries

A. Rantzer's Theorem for Density Functions

For functions $V : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ the notation

$$\nabla V = \left[\begin{array}{ccc} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{array} \right]^T$$

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n}$$

is used. The dual Lyapunov result of [18] is stated as follows:

Theorem 1: Given the equation $\dot{x}(t) = f(x(t))$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f(0) = 0$, suppose there exists a nonnegative function $\rho \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ such that $\rho(x) f(x) / \|x\|$ is integrable on $\{x \in \mathbb{R}^n : \|x\| \geq 1\}$ and

$$[\nabla \cdot (f\rho)](x) > 0 \text{ for almost all } x \tag{3}$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

We shall call ρ a "Rantzer" density function while equation (3) will be called "Rantzer" condition.

B. Decentralized Navigation Functions

In previous work [3], [4], [7] a decentralized navigation functions (DNF's) method for multiple agents with single integrator kinematics was proposed by the authors. In this paper, we redefine the DNF framework of the aforementioned papers in a manner that resembles more the framework of [10] . Specifically, each agent is equipped with a decentralized navigation function $\varphi_i : \mathbb{R}^{2N} \to [0, 1]$ defined as

$$\varphi_i = \frac{\gamma_{di}}{\left(\gamma_{di}^k + G_i\right)^{1/k}} \tag{4}$$

The term $\gamma_{di} = \|q_i - q_{di}\|^2$ is the squared metric of the agent's i configuration from its desired destination q_{di}. The exponent k is a scalar positive parameter. The function G_i expresses the possible collisions of agent i with the others. In particular, G_i is constructed to render the motion produced by the negated gradient of φ_i with respect to q_i repulsive with respect to the other agents. The control law is hence of the form

$$u_i = -K \frac{\partial \varphi_i}{\partial q_i} \tag{5}$$

where $K > 0$ is a positive scalar gain. In this paper, the function G_i is constructed to take into account the limited sensing capabilities of each agent. Using a similar construction with [3],[2] we define the G_i function as

$$G_i = \prod_{\substack{j=0 \\ j \neq i}}^{N} \gamma_{ij}$$

where the function γ_{ij}, for $j = 1, \ldots, N, j \neq i$ is given by

$$\gamma_{ij}\left(\beta_{ij}\right) = \begin{cases} \frac{1}{2}\beta_{ij}, \; 0 \leq \beta_{ij} \leq c^2 \\ \phi(\beta_{ij}), \; c^2 \leq \beta_{ij} \leq d^2 \\ 1, \; d^2 \leq \beta_{ij} \end{cases}$$

where

$$\beta_{ij} = \|q_i - q_j\|^2 - 4r^2$$

is the squared Euclidean distance between agents i and j. The function γ_{i0} refers to the workspace boundary (indexed by 0) and is used to maintain the agents within the workspace. We have

$$\beta_{i0} = (R_W - r)^2 - \|q_i\|^2$$

The function γ_{i0} is defined in the same way as $\gamma_{ij}, j > 0$. The positive constant scalar parameters c, d and the function ϕ are chosen in such a way so that γ_{ij} is everywhere twice continuously differentiable. This is accomplished by choosing an appropriate third degree polynomial function:

$$\phi(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

The parameters of this function are calculated so that γ_{ij} is everywhere twice continuously differentiable. Figure 2 shows a plot of the function γ_{ij} with respect to β_{ij} for $d^2 = 0.96$ and appropriate choice of the other parameters.

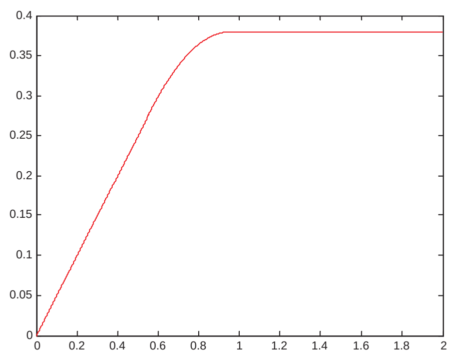

Fig. 2. The function γ_{ij} for $d^2 = 0.96$.

The gradient of φ_i is calculated as

$$\frac{\partial \varphi_i}{\partial q_i} = \frac{\left(\gamma_{di}^k + G_i\right)^{1/k} \nabla_i \gamma_{di} - \frac{\gamma_{di}}{k}\left(\gamma_{di}^k + G_i\right)^{1/k-1}\left(k\gamma_{di}^{k-1}\nabla_i\gamma_{di} + \nabla_i G_i\right)}{\left(\gamma_{di}^k + G_i\right)^{2/k}}$$
$$\Rightarrow \frac{\partial \varphi_i}{\partial q_i} = \left(\gamma_{di}^k + G_i\right)^{-1/k-1}\left(G_i \nabla_i \gamma_{di} - \frac{\gamma_{di}}{k}\nabla_i G_i\right) \tag{6}$$

The construction of the G_i function allows each agent to take into account only agents that are located within T_i at each time instant.

In the sequel, we use the notation $\nabla_i(\cdot) \triangleq \frac{\partial}{\partial q_i}(\cdot)$ for brevity. A critical point of φ_i occurs whenever $\frac{\partial \varphi_i}{\partial q_i} = 0$. The destination point q_{di} is a non-degenerate local minimum of φ_i. The reader is referred to [4] for a proof.

The free space boundary for agent i is defined as the set where $G_i \to 0$. Following the recipe of [10], [4], the next Proposition shows that the negated gradient motion induced by (5) leads to collision avoidance:

Proposition 1: The controller (5) points towards the interior of the free space whenever $G_i \to 0$ for each i.
Proof: At a point q_0 for which $G_i \to 0$, we have

$$\frac{\partial \varphi_i}{\partial q_i}(q_0) = \left(\gamma_{di}^k\right)^{-1/k-1}\left(-\frac{\gamma_{di}}{k}\nabla_i G_i\right)$$

Since the boundary of the free space for agent i is the set where $G_i = 0$, the negated gradient motion $-\frac{\partial \varphi_i}{\partial q_i}$ will point towards the interior of the free space, i.e. towards the set $G_i > 0$. $\diamondsuit$

Since this result holds simultaneously for all agents, collision avoidance is guaranteed. The next result of the current paper, which also follows the procedure of [10], [4], guarantees that the critical points of each navigation function can be constrained to a subset of the state space where G_i is arbitrarily small:

Proposition 2: For every $\varepsilon > 0$ there exists a positive scalar $P > 0$ such that if $k \geq P$ then there are no critical points of φ_i in the set $F_i = \{q \in W | \gamma_{ij} \geq \varepsilon, \forall j \in \mathcal{N}, j \neq i\} \setminus \{\gamma_{di}\}$.
Proof: At a critical point, we have

$$\nabla_i \varphi_i = 0 \Rightarrow G_i \nabla_i \gamma_{di} = \frac{\gamma_{di}}{k}\nabla_i G_i \Rightarrow$$
$$\Rightarrow 2kG_i = \sqrt{\gamma_{di}}\,\|\nabla_i G_i\|$$

since $\|\nabla_i \gamma_{di}\| = 2\sqrt{\gamma_{di}}$. A sufficient condition for this equality not to hold in F_i is given by

$$k > \frac{\sqrt{\gamma_{di}}\,\|\nabla_i G_i\|}{2G_i}, \ \forall q \in F_i.$$

An upper bound for the right hand side is given by

$$\frac{\sqrt{\gamma_{di}}\|\nabla_i G_i\|}{2G_i} \leq \frac{\sqrt{\gamma_{di}}}{2}\sum_{j \neq i}\frac{\|\nabla_i \gamma_{ij}\|}{\gamma_{ij}} \leq$$
$$\frac{1}{2\varepsilon}\max_W\left\{\sqrt{\gamma_{di}}\right\}\sum_{j\neq i}\max_W\left\{\|\nabla_i \gamma_{ij}\|\right\} \triangleq P$$

since $\gamma_{ij} \geq \varepsilon, \forall j \in \mathcal{N}, j \neq i$. Note that the terms $\max_W\left\{\sqrt{\gamma_{di}}\right\}$, $\max_W\left\{\|\nabla_i \gamma_{ij}\|\right\}$ are bounded due to the boundedness of the workspace. $\diamondsuit$

Based on the result of this Proposition, we can choose a sufficiently large k in order to ensure that whenever i has not reached its destination, the critical points of i are located at configurations where $\gamma_{ij} \leq \varepsilon$, for at least one $j \neq i$. By the definition of i, we can choose ε small enough, so that the condition $\gamma_{ij} \leq \varepsilon$ implies $\beta_{ij} \leq 2c^2$, i.e. $\gamma_{ij} = \frac{1}{2}\beta_{ij}$.

IV. Convergence Analysis via Rantzer's Theorem

From (5), it can be deduced that each agent either navigates towards its desired destination avoiding collisions with the others, or converges to a critical point of the corresponding DNF. Specifically, in the DNF framework of [4] we showed that the system converges to a configuration in which $\frac{\partial \varphi_i}{\partial q_i} = 0$ for all $i \in \mathcal{N}$. Using arguments from Morse theory [10], [15], it was then shown that the largest invariant set contained in the set $\frac{\partial \varphi_i}{\partial q_i} = 0$ for all $i \in \mathcal{N}$, is the set of desired destination points, for almost all initial conditions. The proof procedure of [4] does not hold in the approach of the current paper though. The construction of the DNFs in [4] took into account the case when the initial conditions of some agents coincided with their desired destinations. In particular, all agents were forced to participate in the collision avoidance procedure even if their initial state coincided with their desired destination. The reader is referred to the aforementioned paper for more details. In the DNF framework of this paper, it is clear that if $\gamma_{di}=0$ for some agent i at a time instant t_0, then $u_i(t)=0$ for all $t\geq t_0$, i.e. the agent won't participate in the collision avoidance procedure. In essence, the stability analysis of [4] does no longer hold in this case.

In this paper, we examine the invariance of the set $\frac{\partial\varphi_i}{\partial q_i}=0,\forall i\in\mathcal{N}$ via Rantzer's condition (3). It is shown that in the current framework convergence is feasible only to a subset of the state space surrounding the target locations, and not to the target locations themselves.

In the sequel, we denote $\nabla_i(\cdot)\triangleq\frac{\partial}{\partial q_i}(\cdot),\nabla_i^2(\cdot)\triangleq\left\{\frac{\partial^2}{\partial x_i^2}(\cdot),\frac{\partial^2}{\partial y_i^2}(\cdot)\right\}$ for notational thrift.

Specifically, the following Theorem holds:

Theorem 3: Assume that the multi-agent team (1) navigates under the control law (5). Then a sufficient condition for the system to satisfy Rantzer's condition (3) at a stationary point

$$\left\{q\in W|\frac{\partial\varphi_i}{\partial q_i}=0,\ \forall i\in\mathcal{N}\right\}$$

is given by $\gamma_{di}>\gamma_{\min}>0,\forall i\in\mathcal{N}$.

Proof: The closed loop kinematics of system (1) under the control law (5) are given by

$$\begin{gathered}\dot{q}=f(q)=\\ \left[\begin{array}{l}-K\left(\gamma_{d1}^k+G_1\right)^{-1/k-1}\left\{G_1\nabla_1\gamma_{d1}-\frac{\gamma_{d1}}{k}\nabla_1G_1\right\}\\ \vdots\\ -K\left(\gamma_{dN}^k+G_N\right)^{-1/k-1}\left\{G_N\nabla_N\gamma_{dN}-\frac{\gamma_{dN}}{k}\nabla_NG_N\right\}\end{array}\right]\end{gathered}$$

Define $\varphi=\sum\limits_i\varphi_i$ and $\rho=\varphi^{-1}$ and note that ρ is a suitable density function for the equilibrium point q_d. We can then calculate

$$\nabla\rho=-\varphi^{-2}\nabla\varphi$$

and

$$\begin{aligned}&\nabla\cdot(f\rho)=\nabla\rho\cdot f+\rho\nabla\cdot f=\\&=-\varphi^{-2}\nabla\varphi\cdot f+\varphi^{-1}\nabla\cdot f\end{aligned}$$

Whenever $\nabla_i\varphi_i=0$ for all $i\in\mathcal{N}$, we have $f=0$ and

$$\nabla\cdot(f\rho)=\varphi^{-1}\nabla\cdot f=-\varphi^{-1}\sum_iK\left(\frac{\partial^2\varphi_i}{\partial x_i^2}+\frac{\partial^2\varphi_i}{\partial y_i^2}\right)$$

A sufficient condition for the right hand side of the last equation to be strictly positive is

$$\frac{\partial^2\varphi_i}{\partial x_i^2}+\frac{\partial^2\varphi_i}{\partial y_i^2}<0$$

for all $i\in\mathcal{N}$.

Using the notation $\nabla_i^2(\cdot)$ for either $\frac{\partial^2}{\partial x_i^2}(\cdot)$ or $\frac{\partial^2}{\partial y_i^2}(\cdot)$, we have

$$\begin{aligned}&\tfrac{\partial^2\varphi_i}{\partial x_i^2}+\tfrac{\partial^2\varphi_i}{\partial y_i^2}<0\Leftrightarrow\\&G_i\left(\tfrac{\partial^2\gamma_{di}}{\partial x_i^2}+\tfrac{\partial^2\gamma_{di}}{\partial y_i^2}\right)-\tfrac{\gamma_{di}}{k}\left(\tfrac{\partial^2G_i}{\partial x_i^2}+\tfrac{\partial^2G_i}{\partial y_i^2}\right)<0\Leftrightarrow\\&4G_i-\tfrac{\gamma_{di}}{k}\left(\tfrac{\partial^2G_i}{\partial x_i^2}+\tfrac{\partial^2G_i}{\partial y_i^2}\right)<0\end{aligned}$$

since $\nabla_i^2\gamma_{di}=2$ and

$$\begin{aligned}&\nabla_i\varphi_i=0\Rightarrow\\&\nabla_i^2\varphi_i=\left(\gamma_{di}^k+G_i\right)^{-2(1/k+1)}\left(G_i\nabla_i^2\gamma_{di}-\tfrac{\gamma_{di}}{k}\nabla_i^2G_i\right)\end{aligned}$$

Therefore, in order to have $[\nabla\cdot(f\rho)]>0$, it suffices that

$$4G_i-\frac{\gamma_{di}}{k}\left(\frac{\partial^2G_i}{\partial x_i^2}+\frac{\partial^2G_i}{\partial y_i^2}\right)<0\qquad(7)$$

where we stress out again that the notation $\nabla_i^2(\cdot)$ refers to both $\frac{\partial^2}{\partial x_i^2}(\cdot)$ and $\frac{\partial^2}{\partial y_i^2}(\cdot)$.

Using now the notation $\nabla_i^1(\cdot)\triangleq\left\{\frac{\partial}{\partial x_i}(\cdot),\frac{\partial}{\partial y_i}(\cdot)\right\}$ and $\bar{\gamma}_{ij}\triangleq\prod\limits_{k\neq i,j}\gamma_{ik}$ we can compute

$$G_i=\prod_{j\neq i}\gamma_{ij}\Rightarrow\nabla_i^1G_i=\sum_{j\neq i}\bar{\gamma}_{ij}\nabla_i^1\gamma_{ij}$$

and

$$\begin{aligned}\nabla_i^2G_i&=\sum_{j\neq i}\left\{\nabla_i^1\bar{\gamma}_{ij}\nabla_i^1\gamma_{ij}+\bar{\gamma}_{ij}\nabla_i^2\gamma_{ij}\right\}\\&=\sum_{j\neq i}\left\{\nabla_i^1\bar{\gamma}_{ij}\nabla_i^1\gamma_{ij}+\bar{\gamma}_{ij}\right\}\end{aligned}$$

since $\nabla_i^2\gamma_{ij}=1$ for $\beta_{ij}<c^2$. Hence

$$\frac{\nabla_i^2G_i}{G_i}=\sum_{j\neq i}\frac{\nabla_i^1\bar{\gamma}_{ij}\nabla_i^1\gamma_{ij}}{G_i}+\sum_{j\neq i}\frac{1}{\gamma_{ij}}$$

since $\frac{\bar{\gamma}_{ij}}{G_i}=\frac{1}{\gamma_{ij}}$. For $G_i\to0^+$, we have $\gamma_{ij}\to0^+$ for at least one $j\neq i$, by definition of G_i. Hence

$$\frac{1}{\gamma_{ij}}\to+\infty\Rightarrow\frac{\nabla_i^2G_i}{G_i}\to+\infty\Rightarrow\frac{G_i}{\nabla_i^2G_i}\to0^+$$

for $G_i\to0^+$.

By continuity of the function $\frac{\nabla_i^2G_i}{G_i}$ and since $\frac{G_i}{\nabla_i^2G_i}\to0^+$ for $G_i\to0^+$, we conclude that there exists a $M\geq0$, such that $0<\frac{G_i}{\nabla_i^2G_i}\leq M$ for $0<\gamma_{ij}\leq\varepsilon$.

Condition (7) now yields $\gamma_{di}>2kM,\forall i\in\mathcal{N}$. $\diamondsuit$

Some remarks on the condition of Theorem 3 are in order. First of all, it should be pointed out that the condition is far from necessary. Taking into account that k is chosen according to Proposition 2 the lower bound on γ_{di} that is derived is rather conservative. In other words, we show that a finite lower bound γ_{min} exists, but we do not calculate this bound explicitly. Note however that the smaller ϵ is chosen, the smaller M becomes.

It should also be pointed out that the properties imposed on the *scalars* $\frac{\partial^2\varphi_i}{\partial x_i^2},\frac{\partial^2\varphi_i}{\partial y_i^2}$ are not equivalent to the spectral properties of the *matrix* $\frac{\partial^2\varphi_i}{\partial q_i^2}$, from which the Morse properties of the DNF framework are derived in [4]. Clearly, the property on the diagonal elements of the matrix $\frac{\partial^2\varphi_i}{\partial q_i^2}$, as imposed by the condition of Theorem 3, does not imply the sign definiteness of the eigenvalues of this matrix. Please note also that the condition $\frac{\partial^2\varphi_i}{\partial x_i^2}+\frac{\partial^2\varphi_i}{\partial y_i^2}<0,\quad\forall i\in\mathcal{N}$ of Theorem 3 can be replaced by

$$\sum_{i=1}^N\left(\frac{\partial^2\varphi_i}{\partial x_i^2}+\frac{\partial^2\varphi_i}{\partial y_i^2}\right)<0$$

This would result in a less conservative bound for γ_{min}. The analysis is identical to that of Theorem 3.

On the other hand, Theorem 3 justifies the fact that in order to avoid local minima, the agents must be far enough from their targets when they approach a critical point not coinciding with their targets. Since critical points occur whenever agents are near a possible collision (see Proposition 2), it is evident that the DNF framework of the current paper drives the agents to their desired destination only for scenarios where collisions for each agent tend to occur far away from the destination positions. As long as the condition of Theorem 3 holds, agents navigate towards their target location, according to Theorem 1. In essence, the DNF framework of the current paper can only guarantee convergence to a certain subset of the state space containing the target locations, but not to the target locations them selves. The conclusions of this section can be summarized in the following Theorem:

Theorem 4: Assume that the multi-agent team (1) navigates under the control law (5). Then for every initial condition $q(0)$, there a finite strictly positive $\gamma_{\min}$ such that the closed loop system converges to the set

$$\{q \in W | \gamma_{di} \leq \gamma_{\min}, \ \forall i \in \mathcal{N}\}.$$

It is obvious that different values of $\gamma_{\min}$ can be obtained for different relative distances between agents' final and initial conditions. The exact relation of the parameter $\gamma_{\min}$ with the set of initial/final positions is a topic of ongoing research.

V. Simulations

In this section we provide two computer simulations to support the derived conclusions of this paper.

In the first simulation, the case where the sufficient condition of Theorem 3 is clearly violated. In the first screenshot of Figure 3 the initial position and desired destination of agent $i, i = 1, 2, 3, 4$ are denoted by $A-i$,$T-i$ respectively. The parameters of this simulation have been chosen as
Initial Conditions:

$$q_1(0) = \begin{bmatrix} 0 & -.15 \end{bmatrix}^T, q_2(0) = \begin{bmatrix} 0 & .15 \end{bmatrix}^T,$$
$$q_3(0) = \begin{bmatrix} .15 & -.02 \end{bmatrix}^T, q_4(0) = \begin{bmatrix} .02 & 0 \end{bmatrix}^T$$

Final Conditions:

$$q_{d1} = \begin{bmatrix} .1432 & .14 \end{bmatrix}^T, q_{d2} = \begin{bmatrix} - .1732 & -.2 \end{bmatrix}^T,$$
$$q_{d3} = \begin{bmatrix} .1732 & 0 \end{bmatrix}^T, q_{d4} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

Parameters:

$$k = 100, r = .05, d = .11, R_w = 1$$

Screenshots I-IV show the evolution of the multi-agent team in time under the control law (5). We observe that the control design fails to drive the system to the desired equilibria, despite the fact that there is a clear collision-free path for agents 1,2 (note that R_w has been chosen sufficiently large). This is due to the fact that agents 3,4 are very close to their desired destination at the time when a collision between the team members tends to occur. In such a situation, Rantzer's condition does not hold and some of the agents (in particular 1,2) converge to a critical point that does not coincide with their desired destination point.

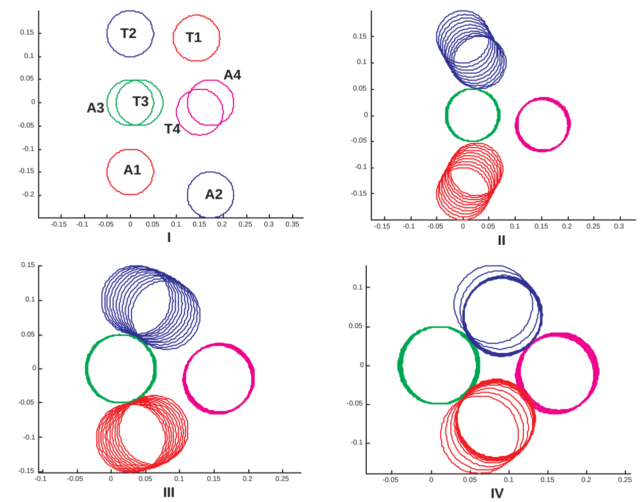

Fig. 3. Agents fail to converge to their desired destinations.

Convergence however takes place in the next simulation that involves seven kinematic agents and a more "collision-bound" scenario. In Figure 4, the seven agents initial and final positions are shown. The initial and final positions, as well as the controller parameters of this simulation have been chosen as:
Initial Conditions:

$$q_1(0) = \begin{bmatrix} -.1 & .86 \end{bmatrix}^T, q_2(0) = \begin{bmatrix} .1 & .096 \end{bmatrix}^T,$$
$$q_3(0) = \begin{bmatrix} .1 & -.045 \end{bmatrix}^T, q_4(0) = \begin{bmatrix} -.1 & -.055 \end{bmatrix}^T,$$
$$q_5(0) = \begin{bmatrix} 0 & -.1 \end{bmatrix}^T, q_6(0) = \begin{bmatrix} 0 & .1 \end{bmatrix}^T,$$
$$q_7(0) = \begin{bmatrix} .18 & 0 \end{bmatrix}^T$$

Final Conditions:

$$q_{d1} = \begin{bmatrix} .1 & -.165 \end{bmatrix}^T, q_{d2} = \begin{bmatrix} .1 & .096 \end{bmatrix}^T,$$
$$q_{d3} = \begin{bmatrix} -.1 & .145 \end{bmatrix}^T, q_{d4} = \begin{bmatrix} .1 & .145 \end{bmatrix}^T,$$
$$q_{d5} = \begin{bmatrix} 0 & .1 \end{bmatrix}^T, q_{d6} = \begin{bmatrix} 0 & -.1 \end{bmatrix}^T,$$
$$q_{d7} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

Parameters:

$$k = 64, r = .035, d = .1, R_w = 1$$

In Figure 5, screenshots I-V show the evolution in time of the seven agents under the control law (5). Although the workspace is more crowded, both the collision avoidance and destination convergence objectives take place. This is due to the fact that agents are far enough from their desired destination at the time when a collision between the team members tends to occur and the conditions of Theorem 3 are not violated.

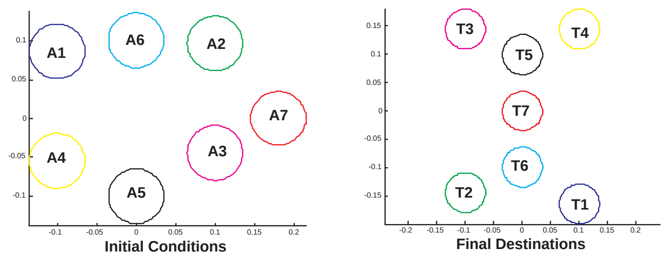

Fig. 4. Initial positions and desired final locations of the seven agents in the second simulation.

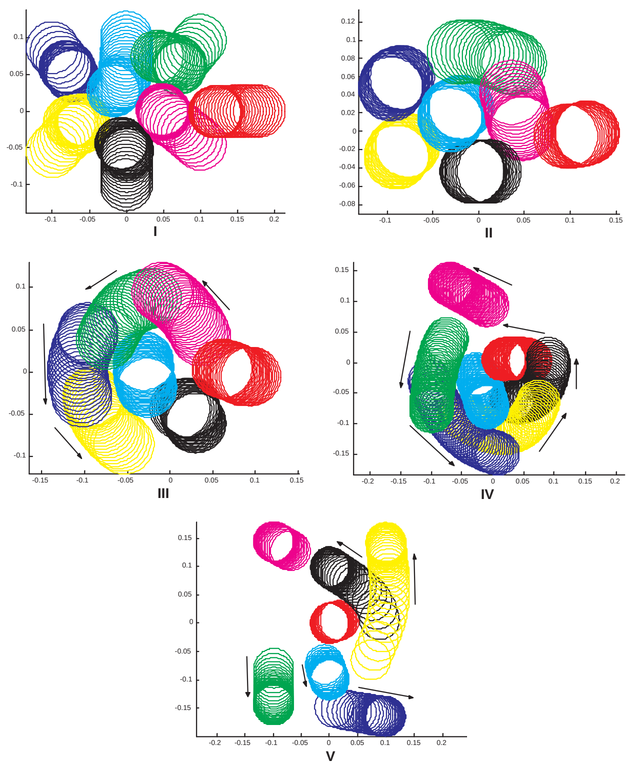

Fig. 5. Agents converge to their desired destinations successfully, avoiding at the same time collisions with each other.

VI. Conclusions

We provided a connection between Rantzer's dual Lyapunov Theorem that appeared in [18] with Decentralized Navigation Functions. It was shown that when the agents' control law does not contain an element that forces them to cooperate with the rest of the team once they have reached their desired goal, global convergence cannot be guaranteed. A sufficient condition for this to happen was derived based on Rantzer's Theorem. In particular, it was shown that agents are driven towards their goals provided that collisions between the team members tend to occur whenever agents are found sufficiently far from their desired destinations. This was derived based on the properties of the critical points of the DNF's imposed by Rantzer's Theorem. The approach of this paper can be considered as a new approach to guaranteed local-minima free decentralized control designs.

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