Family of Controllers for Attitude Synchronization in $\mathcal{S}^2$

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***Abstract*— In this paper we study a family of controllers that guarantees attitude synchronization for a network of elements in the unit sphere domain, i.e., $\mathcal{S}^2$. We propose distributed continuous controllers for elements whose dynamics are controllable (i.e., control with torque as command), and which can be implemented by each individual agent without the need of a common global orientation frame among the network, i.e., it requires only local information that can be measured by each individual agent from its own orientation frame. The controllers are specified according to arbitrary distance functions in $\mathcal{S}^2$, and we provide conditions on those distance functions that guarantee that *i)* a synchronized network of agents is locally asymptotically stable for an arbitrary connected network topology; *ii)* a synchronized network can be achieved for almost all initial conditions in a tree graph network. We also study the equilibria configurations that come with specific types of network graphs. The proposed strategies can be used in attitude synchronization of swarms of fully actuated rigid bodies, such as satellites.**

I. Introduction

Decentralized control in a multi-agent environment has been a topic of active research for the last decade, with applications in large scale robotic systems. Attitude synchronization in satellite formations is one of those applications [1], where the control goal is to guarantee that a network of fully actuated rigid bodies can acquire a common attitude. Coordination of underwater vehicles in ocean exploration missions can also be casted as an attitude synchronization problem [2].

In the literature of attitude synchronization, solutions for consensus in the special orthogonal group can be found [1], [3]–[10], which focus on *complete* attitude synchronization. In this paper, we focus on *incomplete* attitude synchronization, which has not received the same attention: in this scenario each rigid body has a main direction and the global objective is to guarantee alignment of all rigid bodies' main directions; the space orthogonal to each main direction can be left free of actuation or controlled to accomplish some other goals. Complete attitude synchronization requires more measurements when compared to incomplete attitude synchronization, and it might be the case that a rigid body (such as a satellite) is not fully actuated but rather only actuated in the space orthogonal to a specific direction, in which case incomplete attitude synchronization is still feasible.

In [4], attitude control in a leader-follower network of rigid bodies has been studied, with the special orthogonal group being parametrized with Modified Rodrigues Parameters. The proposed solution guarantees attitude synchronization for connected graphs, but it requires all rigid bodies to be aware of a common and global orientation frame. In [5], [6], a controller for a single-leader single-follower network is proposed that guarantees global attitude synchronization at the cost of introducing a discontinuity in the control laws. In [7], attitude synchronization in a leader-follower network is accomplished by designing a non-linear distributed observer for the leader.

In another line of work, in [3], [8], attitude synchronization is accomplished without the need of a common orientation frame among agents. Additionally, in [3], a controller for switching and directed network topologies is proposed, and local stability of consensus in connected graphs is guaranteed, provided that the control gain is sufficiently high. In this paper, we provide a stronger result, by describing the basin of attraction of a synchronized network for a family of controllers.

In [1], attitude synchronization is accomplished with controllers based on behavior based approaches and for a bidirectional ring topology. The special orthogonal group is parametrized with quaternions, and the proposed strategy also requires a common attitude frame among agents. In [11], a quaternion based controller is proposed that guarantees a synchronized network of rigid bodies is a global equilibrium configuration, provided that the graph network is acyclic. This comes at the cost of having to design discontinuous (hybrid) controllers.

In [9], controllers for complete attitude synchronization and for switching topologies are proposed, but this is accomplished at the kinematic level, i.e., by controlling the agents' angular velocity (rather than their torque). This work is extended in [10] by providing controllers at the torque level, and similarly to [1], stability properties rely of high gain controllers.

In this paper, we propose a distributed control strategy for synchronization of elements in the unit sphere domain. The controllers are described as functions of arbitrary distance functions, and, in order to exploit results of graph theory, we impose a condition on those distance functions that will restrict them to be invariant to rotations. As a consequence, the proposed controllers can be implemented by each agent without the need of a common orientation frame. Also, when performing synchronization along a principal axis, we propose a controller that does not require full torque, but rather torque orthogonal to that principal axis. We restrict the proposed controllers to be continuous, which means that a synchronized network of agents cannot be a global equilibrium configuration, since $\mathcal{S}^2$ is a non-contractible

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set [12]. Our main contribution lies in finding conditions on the distance functions that guarantee that *i)* a synchronized network is locally asymptotically stable for arbitrary connected network topologies; *ii)* a synchronized network can be achieved for almost all initial conditions in a tree graph network. Also, we provide explicit basins of attraction of the synchronized network, and characterize the equilibria configurations for some general, yet specific, types of graph networks. The remainder of this paper is structured as follows. In Section III, the problem statement is described; in Section IV, the proposed solution is presented; in Sections V and VI, convergence to a synchronized network is discussed for tree and non-tree graphs, respectively; in Section VII an alternative constrained control law is proposed; and, in Section VIII, simulations are presented that support the theoretical results.

II. Notation

Given a matrix $A \in \mathbb{R}^{n\times m}$, $(A)_j$ denotes the j^{th} column of A for all $j \in \{1,\cdots,m\}$, and $(A^T)_i^T$ denotes the i^{th} row of A for all $i \in \{1,\cdots,n\}$. $\mathbf{0}_n \in \mathbb{R}^n$ and $\mathbf{1}_n \in \mathbb{R}^n$ denote the zero column vector and the column vector with all components equal to 1, respectively; when the subscript $_n$ is omitted, the dimension n is assumed to be of appropriate size. $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ stands for the identity matrix, and we omit its subscript when $n=3$. The matrix $\mathcal{S}(\mathbf{x}) \in \mathbb{R}^{3\times 3}$ is a cross product skew-symmetric matrix and it satisfies $\mathcal{S}(\mathbf{a})\,\mathbf{b} = \mathbf{a}\times\mathbf{b}$, for any $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$. The map $\Pi(\mathbf{x}) : \{\mathbf{x}\in\mathbb{R}^3 : \mathbf{x}^T\mathbf{x} = 1\} \mapsto \mathbb{R}^{3\times 3}$ yields a matrix that represents the orthogonal projection operator onto the subspace perpendicular to $\mathbf{x}$, and it satisfies $\Pi(\mathbf{x}) = \mathbf{I} - \mathbf{x}\mathbf{x}^T$. We denote the Kronecker product between $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{s\times t}$ by $A\otimes B \in \mathbb{R}^{m\,s\times n\,t}$. For a function $f(.)$ parametrized in time, we denote $f|_{t=t'}$ as f evaluated at time instant t'. For two vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$, $\mathbf{a} = \pm\mathbf{b}$ means that either $\mathbf{a}=\mathbf{b}$ or $\mathbf{a}=-\mathbf{b}$. For $\mathbf{e}\in\mathbb{R}^n$, e_i stands for the i^{th} component; and for $\mathbf{e}\in\mathbb{R}^{Mn}$, $\mathbf{e}_i \triangleq \begin{bmatrix} e_{M(i-1)+1} & \cdots & e_{Mi}\end{bmatrix}^T$ ($n, M \in \mathbb{N}$). We also use the generic notation $\{\mathbf{a}_i\} \triangleq \{\mathbf{a}_1,\cdots,\mathbf{a}_N\}$ where $\mathbf{a}_i \in \mathcal{A}$ for all $i\in\{1,\cdots,N\}$ and some $N\in\mathbb{N}$, and where $\mathcal{A}$ is an arbitrary space (for example, $\mathcal{S}^2$ or $\mathbb{R}^3$ or $\mathbb{N}$); finally, we denote $|\{\mathbf{a}_i\}| = N$.

III. Problem Statement

We consider a group of N agents, indexed by the set $\mathcal{N} = \{1,\cdots,N\}$, operating in the unit sphere domain $\mathcal{S}^2 = \{\mathbf{x}\in\mathbb{R}^3 : \mathbf{x}^T\mathbf{x} = 1\}$. The group of agents network can be modeled as an undirected static graph, $\mathcal{G} = \{\mathcal{N},\mathcal{E}\}$, with $\mathcal{N}$ as the vertices' set indexed by the team members, and $\mathcal{E}$ as the edges' set. For every pair of agents, $i\in\mathcal{N}$ and $j\in\mathcal{N}\backslash\{i\}$, that are aware of each other's relative attitude, we say agent j is a neighbor of i, and vice-versa; also, we denote $\mathcal{N}_i \subset \mathcal{N}$ as the neighbor set of agent i. Each agent i has its own orientation frame (w.r.t. an unknown inertial orientation frame), represented by $\mathcal{R}_i \in \mathcal{SO}(3)$. Let $\mathbf{n}_i \in \mathcal{S}^2$ be a direction along agent's i orientation, i.e., $\mathbf{n}_i \triangleq \mathcal{R}_i\bar{\mathbf{n}}_i$, where $\bar{\mathbf{n}}_i \in \mathcal{S}^2$ is a constant unit vector that is known by agent i and no other agent. In this paper, the goal of attitude synchronization is not that all agents share the same *complete* orientation, i.e., $\mathcal{R}_1 = \cdots = \mathcal{R}_N$, but rather that all agents share the same orientation along a specific direction, i.e., $\mathbf{n}_1 = \cdots = \mathbf{n}_N$. Figure 1 illustrates the concept of incomplete synchronization for two agents. Notice that agent i is not aware of $\mathbf{n}_i$ (since this is specified in an unknown inertial orientation frame); instead, agent i is aware of its direction $\bar{\mathbf{n}}_i$ – fixed in its own orientation frame – and the relative attitude between its direction and its neighbors' own directions. For example, in a group of satellites that must align one of their principal axis – say the first axis, $\bar{\mathbf{n}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ for all satellites, and the desired synchronized network of satellites satisfies $\mathcal{R}_1\bar{\mathbf{n}} = \cdots = \mathcal{R}_N\bar{\mathbf{n}}$.

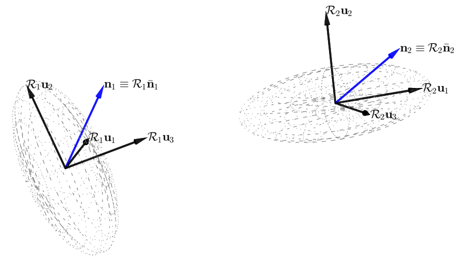


Fig. 1. In incomplete synchronization, n rigid bodies, indexed by $i = \{1,\cdots,n\}$, align the unit vectors $\mathbf{n}_i \triangleq \mathcal{R}_i\bar{\mathbf{n}}_i$, where $\bar{\mathbf{n}}_i$ is fixed in rigid body i ($\mathbf{u}_1$,$\mathbf{u}_2$ and $\mathbf{u}_3$ stand for the canonical basis vectors in $\mathbb{R}^3$).

A rotation matrix $\mathcal{R}\in\mathcal{SO}(3)$ evolves with kinematics

$$\dot{\mathcal{R}} = \mathcal{R}\mathcal{S}(\boldsymbol{\omega}) \Rightarrow \dot{\mathcal{R}}\bar{\mathbf{n}} = \mathcal{S}(\mathcal{R}\boldsymbol{\omega})\,\mathcal{R}\bar{\mathbf{n}}, \tag{1}$$

where $\boldsymbol{\omega}\in\mathbb{R}^3$ is the body-framed angular velocity. For a rigid body with moment of inertia $J = J^T \in \mathbb{R}^{3\times 3}$, the body-framed angular velocity dynamics are given by

$$J\dot{\boldsymbol{\omega}} = -\mathcal{S}(\boldsymbol{\omega})\,J\boldsymbol{\omega} + \mathbf{T}, \tag{2}$$

with $\mathbf{T}\in\mathbb{R}^3$ being a torque expressed in the body attitude frame, and that can be actuated.

Problem 1: Given a set of dynamic agents with unit vectors $\{\mathbf{n}_i\}$, angular velocities $\{\boldsymbol{\omega}_i\}$ and moments of inertia $\{J_i\}$ satisfying (1) and (2), design distributed control laws for the torques $\{\mathbf{T}_i\}$ that guarantee that all unit vectors converge to each other, in the absence of a common inertial orientation frame.

IV. Proposed Solution

A. Preliminaries

We first present some definitions and results from graph theory that will be used in later sections [13]. A graph $\mathcal{G} = \{\mathcal{N},\mathcal{E}\}$ is said to be connected if there exists a path between any two vertices in $\mathcal{N}$. $\mathcal{G}$ is a tree if it is connected and it contains no cycles. An orientation on the graph $\mathcal{G}$ is the assignment of a direction to each edge $(i,j)\in\mathcal{E}$, where each edge vertex is either the tail or the head of the edge. For brevity, we denote $N = |\mathcal{N}|$, $M = |\mathcal{E}|$ and $\mathcal{M} \triangleq \{1,\cdots,M\}$. The incidence matrix $B\in\mathbb{R}^{N\times M}$ of $\mathcal{G}$ is the $\{0,\pm1\}$ matrix, such that $B_{ij} = 1$ if the vertex i is the head of the edge j, $B_{ij} = -1$ if the vertex i is the tail of the edge j, and 0 otherwise. For notational convenience in the analysis that follows, consider the sets $\mathcal{E} = \{(i,j)\in\mathcal{N}\times\mathcal{N} : j\in\mathcal{N}_i\}$, i.e., the set of edges of the graph $\mathcal{G}$; and $\bar{\mathcal{E}} = \{(i,j) \in \mathcal{E} : j > i\}$. For undirected graph networks, we can construct an injective function $\bar{\kappa} : \bar{\mathcal{E}} \mapsto \mathcal{M}$ from which it is possible to construct a second, now surjective, function $\kappa : \mathcal{E} \mapsto \mathcal{M}$, which satisfies $\kappa(i,j) = \bar{\kappa}(i,j)$ when $j > i$ and $\kappa(i,j) = \bar{\kappa}(j,i)$ when $j < i$. As such, by construction, for every $(i,j) \in \mathcal{E}$, $\kappa(i,j) = \kappa(j,i)$, since we consider undirected graphs. The function $\kappa(.,.)$ thus assigns an edge index to every unordered pair of neighbors $\{i,j\}$.

Proposition 1: If $\mathcal{G}$ is a tree, then $B^T B$ is positive definite [14]. The same conclusion holds for $(B \otimes \mathbf{I})^T (B \otimes \mathbf{I})$.

Proposition 2: If $\mathcal{G}$ is connected but not a tree, then the null space of the incidence matrix, i.e., $\mathcal{N}(B)$, is non-empty, and it corresponds to the cycle space of each cycle [15].

When $m \geq 3$ edges form a cycle, we denote $C \subseteq \mathcal{M}$ as a set of indexes that correspond to the cycle edges. We say two cycles C_1 and C_2 are independent if $C_1 \cap C_2 = \emptyset$. We say that two cycles C_1 and C_2 share one edge when $|C_1 \cap C_2| = 1$ and $C_1 \cup C_2$ contains edges from only three cycles: C_1, C_2 and $C = C_1 \cup C_2 \backslash \{C_1 \cap C_2\}$, with $|C| = |C_1| + |C_2| - 2$.

Proposition 3: If $\mathcal{G}$ contains only – say m – independent cycles, then the null space of B is given by $\mathcal{N}(B) = \{\mathbf{e} \in \mathbb{R}^M : e_k = \pm e_l, \forall k, l \in C_i, i = \{1, \cdots, m\}\}$; and the null space of $B \otimes \mathbf{I}_n$ is given by $\mathcal{N}(B \otimes \mathbf{I}_n) = \{\mathbf{e} \in \mathbb{R}^{Mn} : \mathbf{e}_k = \pm \mathbf{e}_l, \forall k, l \in C_i, i = \{1, \cdots, m\}\}$ [16].

A description of the null space of the incidence matrix (similar to that in Proposition 3) for graphs with independent cycles and/or cycles that share only one edge can be found in [16].

B. Distance in $\mathcal{S}^2$

Consider an arbitrary distance function between unit vectors $d(\mathbf{n}_1, \mathbf{n}_2) : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathbb{R}_0^+$, satisfying $d(\mathbf{n}_1, \mathbf{n}_2) \geq 0$; $d(\mathbf{n}_1, \mathbf{n}_2) = 0 \Leftrightarrow \mathbf{n}_1 = \mathbf{n}_2$; and $d(\mathbf{n}_1, \mathbf{n}_2) = d(\mathbf{n}_2, \mathbf{n}_1)$.

We want to exploit the results in Propositions 1-3, which is why we impose the condition

$$\mathcal{S}(\mathbf{n}_1) \frac{\partial d(\mathbf{n}_1, \mathbf{n}_2)}{\partial \mathbf{n}_1} = -\mathcal{S}(\mathbf{n}_2) \frac{\partial d(\mathbf{n}_1, \mathbf{n}_2)}{\partial \mathbf{n}_2}, \tag{3}$$

where $\frac{\partial d(\mathbf{n}_1, \mathbf{n}_2)}{\partial \mathbf{n}_1}$ is the gradient of the distance function w.r.t. the first argument. By imposing such condition, it follows that

$$\dot{d}(\mathbf{n}_1, \mathbf{n}_2) = \begin{bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{bmatrix}^T \begin{bmatrix} \mathcal{R}_1^T & \mathbf{0} \\ \mathbf{0} & \mathcal{R}_2^T \end{bmatrix} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \mathbf{I} \right) \frac{\partial d(\mathbf{n}_1, \mathbf{n}_2)}{\partial \mathbf{n}_1} \tag{4}$$

where we find an incidence matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ corresponding to an edge between unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. In later sections, results from graph theory, such as those presented in Section IV-A, are explored in order to infer properties of the network behavior.

By invoking uniqueness of solutions of the PDE (3), and by verifying that $d(\mathbf{n}_1, \mathbf{n}_2) = f(\arccos(\mathbf{n}_1^T \mathbf{n}_2))$ satisfies (3), one can conclude $f(\arccos(\mathbf{n}_1^T \mathbf{n}_2))$ is the only type of distance function that satisfies (3), for any $f(x) : \begin{bmatrix} 0 & \pi \end{bmatrix} \to \mathbb{R}_0^+$. As such, we restrict ourselves to distance functions of the type $d(\mathbf{n}_1, \mathbf{n}_2) = f(\arccos(\mathbf{n}_1^T \mathbf{n}_2))$, which are invariant to rotation of their arguments, i.e., $d(\mathcal{R}\mathbf{n}_1, \mathcal{R}\mathbf{n}_2) = d(\mathbf{n}_1, \mathbf{n}_2)$ for any $\mathcal{R} \in \mathcal{SO}(3)$. This property will guarantee that the proposed controllers can be implemented without the need of a common inertial orientation frame. A simple distance function is found by choosing $f(x) = x$, in which case we define $d_\theta(\mathbf{n}_1, \mathbf{n}_2) : \mathcal{S}^2 \times \mathcal{S}^2 \to [0, \pi]$ as $d_\theta(\mathbf{n}_1, \mathbf{n}_2) = \arccos(\mathbf{n}_1^T \mathbf{n}_2)$, and whose gradient is given by

$$\frac{\partial d_\theta(\mathbf{n}_1, \mathbf{n}_2)}{\partial \mathbf{n}_1} = \frac{-1}{\sqrt{1 - (\mathbf{n}_1^T \mathbf{n}_2)^2}} \mathbf{n}_2 = \frac{-1}{\|\mathcal{S}(\mathbf{n}_1) \mathbf{n}_2\|} \mathbf{n}_2. \tag{5}$$

The gradient of an arbitrary distance function, $d(\mathbf{n}_1, \mathbf{n}_2) = f(d_\theta(\mathbf{n}_1, \mathbf{n}_2))$, can be obtained from (5) as

$$\frac{\partial d(\mathbf{n}_1, \mathbf{n}_2)}{\partial \mathbf{n}_1} = -\frac{f'(d_\theta(\mathbf{n}_1, \mathbf{n}_2))}{\|\mathcal{S}(\mathbf{n}_1) \mathbf{n}_2\|} \mathbf{n}_2 \triangleq -g(d_\theta(\mathbf{n}_1, \mathbf{n}_2)) \mathbf{n}_2. \tag{6}$$

We will focus on arbitrary distance functions $d(\mathbf{n}_1, \mathbf{n}_2)$ that can be obtained from $d_\theta(\mathbf{n}_1, \mathbf{n}_2)$ by means of an increasing function $f(.)$. As such, it follows that the distance between unit vectors is maximum when two unit vectors are diametrically opposed, i.e., $d(\mathbf{n}_1, -\mathbf{n}_1) = \max_{\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{S}^2} d(\mathbf{v}_1, \mathbf{v}_2) \triangleq d^{\max}, \forall \mathbf{n}_1 \in \mathcal{S}^2$.

Definition 1: Consider a continuous function $g(x) : [0, \pi] \to \mathbb{R}_0^+$ that satisfies $0 < g(x) < +\infty$ for all $x \in (0, \pi)$ and $\lim_{x \to 0^+} g(x) = g^0 < \infty$ and $\lim_{x \to \pi^-} g(x) = g^\pi > 0$. We say $g(x)$ is of

- class $\mathcal{P}$ if $g^0 > 0$ and $g^\pi < +\infty$,
- class $\mathcal{P}^\infty$ if $g^0 > 0$ and $g^\pi = +\infty$,
- class $\mathcal{P}^0$ if $g^0 = 0$ and $g^\pi < +\infty$,
- class $\mathcal{P}^{0,\infty}$ if $g^0 = 0$ and $g^\pi = +\infty$,
- class $\bar{\mathcal{P}}$ if it is of any of the previous classes.

In what follows, all functions introduced can depend on the edge index of the network graph, given by the function $\kappa : \mathcal{E} \mapsto \mathcal{M}$, i.e., (possibly) different distance functions are assigned to different edges. For each edge $k = \kappa(i,j) = \kappa(j,i) \in \mathcal{M}$, we use the notation $d_k(.,.) = d_{\kappa(i,j)}(.,.) = d_{\kappa(j,i)}(.,.)$ interchangeably; also, we denote $d_k^{\max}$ as the maximum of $d_k(.,.)$; $f_k(.)$ as the function $f(.)$ associated to $d_k(.,.)$; and $g_k(.)$ as the function $g(.)$ associated to $d_k(.,.)$. For all $k \in \mathcal{M}$, $g_k(.)$ is of class $\bar{\mathcal{P}}$.

C. Solution to Problem 1

Recall, from Section IV-A, that $\kappa(i,j)$ stands for the edge formed by agents i and j. In our framework, since the distance function can depend on the edge, $d_k(.,.)$ stands for the distance function on edge $k = \kappa(i,j) = \kappa(j,i)$. For edge k, where $\bar{\kappa}^{-1}(k) = (i,j)$, we denote its tail by $k_1 = i$, and its head by $k_2 = j$. In order to accomplish the goal in Problem 1, we propose the following decentralized control law for $\mathbf{T}_i$,

$$\mathbf{T}_i = -\boldsymbol{\sigma}(\boldsymbol{\omega}_i) - \mathcal{R}_i^T \sum\nolimits_{j \in \mathcal{N}_i} \mathcal{S}(\mathbf{n}_i) \frac{\partial d_{\kappa(i,j)}(\mathbf{n}_i, \mathbf{n}_j)}{\partial \mathbf{n}_i}, \tag{7}$$

with $\boldsymbol{\sigma}(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ as a direction preserving function with possibly bounded norm, i.e. $\|\boldsymbol{\sigma}(.)\| \leq \sigma^{\max}$, where $\|\boldsymbol{\sigma}(\mathbf{x})\| \leq \sigma_s \|\mathbf{x}\|$ with $\sigma_s < \infty$; also, $D\boldsymbol{\sigma}(\mathbf{x}) \triangleq \frac{\partial \boldsymbol{\sigma}(\mathbf{x})}{\partial \mathbf{x}}$ is upper bounded by $\|D\boldsymbol{\sigma}(.)\| \leq \sigma'^{\max} < \infty$ (i.e., $\boldsymbol{\sigma}(\mathbf{x})$ is Lipschitz). If we stack all the torque vectors $\mathbf{T}_i$, as $\mathbf{T} \triangleq [\mathbf{T}_1 \cdots \mathbf{T}_N]^T$, we can rewrite (7) as

$$\mathbf{T} = -[\boldsymbol{\sigma}(\boldsymbol{\omega}_i) \cdots \boldsymbol{\sigma}(\boldsymbol{\omega}_N)]^T - \boldsymbol{\mathcal{R}}^T (B \otimes \mathbf{I}) \mathbf{e}, \tag{8}$$

where $\boldsymbol{\mathcal{R}}$ is a block diagonal matrix with rotation matrices $\mathcal{R}_1$ to $\mathcal{R}_N$; and $\mathbf{e} \triangleq [\mathbf{e}_1 \cdots \mathbf{e}_M]^T$ where $\mathbf{e}_k \triangleq \mathcal{S}(\mathbf{n}_{k_1}) \frac{\partial d_k(\mathbf{n}_{k_1}, \mathbf{n}_{k_2})}{\partial \mathbf{n}_{k_1}}$ stands for the error associated to edge $k \in \mathcal{M}$, and $\mathbf{e}$ is the stack variable of all edge errors. As such, from (6), it follows

$$\|\mathbf{e}_k\| = g_k(d_\theta(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}))) \|\mathcal{S}(\mathbf{n}_{k_1}) \mathbf{n}_{k_2}\|, \tag{9}$$

which means this norm can grow unbounded only if two neighbor unit vectors are diametrically opposed, since g_k is of class $\bar{\mathcal{P}}$. Moreover, if g_k is of class $\mathcal{P} \cup \mathcal{P}^0$, then $\|\mathbf{e}_k\| < \infty$, and additionally $\|\mathbf{e}_k\| = 0$ if and only if two neighbor unit vectors are aligned or diametrically opposed.

The proposed torque (7) exhibits three properties worth emphasizing. First, notice that $\mathcal{R}_i^T \mathcal{S}(\mathbf{n}_i) \frac{\partial d_{\kappa(i,j)}(\mathbf{n}_i, \mathbf{n}_j)}{\partial \mathbf{n}_i} = g_{\kappa(i,j)}(d_\theta(\bar{\mathbf{n}}_i, \mathcal{R}_i^T \mathbf{n}_j)) \mathcal{S}(\bar{\mathbf{n}}_i) \mathcal{R}_i^T \mathbf{n}_j$, where $\mathcal{R}_i^T \mathbf{n}_j$ can be measured by agent i in its own reference frame. This means the control law (7) can be implemented in the absence of a common orientation frame among agents. Secondly, if g_k is of class $\mathcal{P}^\infty \cup \mathcal{P}^{0,\infty}$ for some $k \in \mathcal{M}$, the control law may be ill defined and care must be taken in those cases, as is done in Theorem 4. Finally, notice that $\mathbf{T}_i$ can be bounded by $\|\mathbf{T}_i\| \leq \sigma^{\max} + |\mathcal{N}_i| \max_{j \in \mathcal{N}_i} (\max_{0 \leq \theta \leq \pi} g_{\kappa(i,j)}(\theta))$ (if $g_{\kappa(i,j)}$ is an increasing function, then $\max_{0 \leq \theta \leq \pi} g_{\kappa(i,j)}(\theta) = g^\pi_{\kappa(i,j)}$). As such, the proposed control law, for each agent i, can be implemented with bounded actuation provided that $\sigma^{\max} < \infty$, and that all $g_{\kappa(i,j)}$ are of class $\mathcal{P} \cup \mathcal{P}^0$ for all $j \in \mathcal{N}_i$.

For the rest of this paper, we dedicate efforts in studying the equilibria configurations induced by this control law (for different types of graphs), their stability (or lack thereof), and what is the effect of the chosen distance function.

D. Lyapunov Function

All results that follow are based on the same Lyapunov function. It is defined as follows,

$$V = \sum_{k=1}^{M} d_k(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}) + \sum_{l=1}^{N} \frac{1}{2} \boldsymbol{\omega}_l^T J_l \boldsymbol{\omega}_l, \tag{10}$$

where $H \triangleq \sum_{l=1}^N \frac{1}{2} \boldsymbol{\omega}_l^T J_l \boldsymbol{\omega}_l$ stands for the total rotational kinetic energy of the network. The Lyapunov time derivative can be computed, and it yields $\dot{V} = \boldsymbol{\omega}^T \boldsymbol{\mathcal{R}}^T (B \otimes \mathbf{I})\mathbf{e} + \sum_{l=1}^N \boldsymbol{\omega}_l^T \mathbf{T}_l$, where we have used property (3) (see similarity between $\boldsymbol{\omega}^T \boldsymbol{\mathcal{R}}^T (B \otimes \mathbf{I})\mathbf{e}$ and (4)). When we compose $\dot{V}$ with the proposed control law, it follows

$$\dot{V} = -\sum_{l=1}^{N} \boldsymbol{\omega}_l^T \boldsymbol{\sigma}(\boldsymbol{\omega}_l) \leq 0. \tag{11}$$

The second time derivative of V can also be computed, and it yields $\ddot{V} = -\sum_{l=1}^N \left(\boldsymbol{\sigma}^T(\boldsymbol{\omega}_l) + \boldsymbol{\omega}_l^T D\boldsymbol{\sigma}(\boldsymbol{\omega}_l)\right) \dot{\boldsymbol{\omega}}_l$, which can be upper bounded by $\|\ddot{V}\| \leq (\sigma_s + \sigma'^{\max}) \sum_{l=1}^N \frac{\|\boldsymbol{\omega}_l\|}{\lambda_{\min}(J_l)} \left(\|\mathbf{T}_l\| + \lambda_{\max}(J_l) \|\boldsymbol{\omega}_l\|^2\right)$. Finally, notice that $\|\mathbf{T}_l\| \leq \sigma_s \|\boldsymbol{\omega}_l\| + \|B \otimes \mathbf{I}\| \|\mathbf{e}\|$. Thus, $\|\ddot{V}\|$ is upper bounded provided that both $\|\boldsymbol{\omega}_l\|$ and $\|\mathbf{e}\|$ are bounded.

V. Tree Graphs

Let us focus first on static tree graphs. For these graphs, we can invoke Proposition 1 and conclude that the null space of $B \otimes \mathbf{I}$ must be the empty set. For brevity, in what follows we denote $\min_{k \in \mathcal{M}}(d_k^{\max}) \triangleq \min(d_k^{\max})$.

Theorem 4: Consider a static tree topology for a group of unit vectors with kinematics (1) and dynamics (2). Also, consider the control law (8), where g_k is of class $\bar{\mathcal{P}}$ for all $k \in \mathcal{M}$. If $\frac{H|_{t=0}}{\min(d_k^{\max})} < 1$ and

$$d_k(\mathbf{n}_{k_1}, \mathbf{n}_{k_2})|_{t=0} < M^{-1} \left(\min(d_k^{\max}) - H|_{t=0}\right), \tag{12}$$

for all edges k, then all unit vectors converge to each other. If additionally $\min(d_k^{\max}) = \infty$, then all unit vectors converge to each other for almost all initial conditions.

Proof: Consider the Lyapunov function given in Section IV-D. Under the conditions of the Theorem, $V(0) < \min(d_k^{\max})$, and given that $\dot{V}$ is non-positive, it follows that $V(t) < \min(d_k^{\max})$ for all time, which implies *i)* that $H(t)$ is bounded; *ii)* that no distance function can ever reach its maximum and therefore two neighbor unit vectors will never be diametrically opposed. In turn, this guarantees that $\|\mathbf{e}_k\|$ is bounded (see (9)) and so is $\ddot{V}$. Since V is lower bounded (by 0), $\dot{V}$ is non-positive and uniformly continuous, it follows from Barbalat's lemma [17] that $\dot{V}$ must asymptotically converge to 0. This implies that $\boldsymbol{\omega}_i$ converges to $\mathbf{0}$ for all agents (see (11)). Invoking uniform continuity of $\dot{\boldsymbol{\omega}}_i$ ($\mathbf{T}_i$ is bounded under the Theorem's conditions), it follows from (2) that $\mathbf{T}_i$ must converge to zero, and consequently $(B \otimes \mathbf{I})\mathbf{e}$ must also converge to $\mathbf{0}$ (see (7)). For a tree graph, $\mathcal{N}(B \otimes \mathbf{I}) = \emptyset$, which implies that $\mathbf{e}$ converges asymptotically to $\mathbf{0}$. As such, and since two neighbor unit vectors will never be diametrically opposed, it follows that all neighbor unit vectors converge to each other (see (9)). In a connected graph, this means all unit vectors converge to each other. If additionally $\min(d_k^{\max}) = \infty$, condition (12) is satisfied as long as two neighbor unit vectors are not initially diametrically opposed. This corresponds to a set of zero measure in the space of all initial conditions. ■

Condition $\frac{H|_{t=0}}{\min(d_k^{\max})} < 1$ represents an upper bound on the initial rotational kinetic energy, for which we can guarantee that all unit vectors converge to each other. By making $\min(d_k^{\max})$ sufficiently large, convergence can still be guaranteed for arbitrarily large initial rotational energy $H|_{t=0}$, thus enlarging the region of stability, and yielding the almost global stability result for $\min(d_k^{\max}) = \infty$.

Some example distance functions and their properties can be found on [16]. Also, an additional Theorem can be found in [16], which quantifies how many pairs of neighboring unit vectors can be asymptotically diametrically opposed.

VI. Non-Tree Graphs

In this section, we study the equilibria configuration induced by some more general, yet specific, graph topologies, and the local stability properties of the synchronized configuration for arbitrary graphs. We first give the following definition.

Definition 2: Given n vectors $\mathbf{x}_i \in \mathbb{R}^3$, for $i \in \{1, \cdots, n\}$, we say $\{\mathbf{x}_i\}$ *belong to a common plane* if there exists a unit vector $\boldsymbol{\nu} \in \mathcal{S}^2$ such that $\Pi(\boldsymbol{\nu}) \mathbf{x}_i = \mathbf{x}_i$ for all $i \in \{1, \cdots, n\}$.

Proposition 5: Consider the unit vectors $\mathbf{n}_1, \cdots, \mathbf{n}_n$, with $|\mathbf{n}_i^T \mathbf{n}_{i+1}| \neq 1$ for all $i = \{1, \cdots, n-1\}$. If $\frac{\mathcal{S}(\mathbf{n}_1)\mathbf{n}_2}{\|\mathcal{S}(\mathbf{n}_1)\mathbf{n}_2\|} = \cdots = \pm\frac{\mathcal{S}(\mathbf{n}_{n-1})\mathbf{n}_n}{\|\mathcal{S}(\mathbf{n}_{n-1})\mathbf{n}_n\|}$, then all unit vectors belong to a common plane.

A proof on Proposition 5 can be found in [16].

Theorem 6: Consider a group of unit vectors with kinematics (1) and dynamics (2). Also, consider the control law (8), where g_k is of class $\mathcal{P} \cup \mathcal{P}^0$ for all $k \in \mathcal{M}$. If the topology contains only independent cycles, then for each cycle, all its unit vectors converge to a common plane.

Proof: Following the same steps as in the proof of Theorem 4, we conclude that $\mathbf{e}$ must converge to the null space of $B \otimes \mathbf{I}$. Now, consider a graph with only independent cycles and recall Proposition 3. Without loss of generality, consider the first $n \geq 3$ edges are part of a cycle. Two possibilities exist: *i)* $\mathbf{e}$ converges to $\mathbf{0}$ and all unit vectors are either all aligned or some are diametrically opposed to others (in either case all unit vectors converge to a common plane); *ii)* $\mathbf{e}$ converges to some non-zero vector that, due to the network topology, belongs to $\mathcal{N}(B \otimes \mathbf{I})$. From Proposition 3, it follows that $\pm\mathbf{e}_1 = \cdots = \pm\mathbf{e}_n$. In particular, all edges must have the same direction, which implies that $\mathcal{S}(\mathbf{n}_{1_1})\mathbf{n}_{1_2}/\|\mathcal{S}(\mathbf{n}_{1_1})\mathbf{n}_{1_2}\| = \cdots = \pm\mathcal{S}(\mathbf{n}_{n_1})\mathbf{n}_{n_2}/\|\mathcal{S}(\mathbf{n}_{n_1})\mathbf{n}_{n_2}\|$. From Proposition 5, it follows that all unit vectors that form a cycle belong to a common plane. ■

Proposition 6 can be extended for graphs with independent cycles and/or cycles that share only one edge [16].

We now present a proposition, which will be useful in guaranteeing local asymptotic stability of attitude synchronization for arbitrary graphs.

Definition 3: We say that a group of unit vectors belongs to an open α-cone, for $\alpha \in [0, \frac{\pi}{2}]$, if the inner product between any two vectors is strictly larger than $\cos(\alpha)$.

Proposition 7: Consider a set of unit vectors $\{\mathbf{n}_i\}$, for $i \in \mathcal{N} = \{1, \cdots, N\}$, contained in an open $\frac{\pi}{2}$-cone, and assume that *i)* the network graph is connected; *ii)* $\mathbf{e} \in \mathcal{N}(B \otimes \mathbf{I})$, where $\mathbf{e}$ has been defined in (8). This takes place if and only if $\mathbf{n}_i = \mathbf{n}_j$ for all $i, j \in \mathcal{N}$.

A complete proof for Proposition 7 can be found in [16]. In short, if one assumes the unit vectors are not synchronized and that if there exits an $\mathbf{e} \neq \mathbf{0}$ (see (8)) such that $(B \otimes \mathbf{I})\mathbf{e} = \mathbf{0}$, one can find a unit vector $\mathbf{n}_i$, for which $((B^T)_i^T \otimes \mathbf{I})\mathbf{e} \neq \mathbf{0}$, thus reaching a contradiction.

Before we introduce our final result, let us define the quantity $d^\star = \min_k f_k\left(\min_k f_k^{-1}\left(\min_k f_k\left(\frac{\pi}{2}\frac{1}{N-1}\right)\right)\right) \neq 0$, representing a critical distance between two neighbor unit vectors, satisfying $f_k^{-1}(d^\star) \leq \frac{\pi}{2}\frac{1}{N-1}$ for all k. This quantity is well defined, since all $f_k(.)$ are increasing functions.

Theorem 8: Consider an arbitrary graph topology for a group of unit vectors with kinematics (1) and dynamics (2). Also, consider the control law (8), where g_k is of class $\bar{\mathcal{P}}$ for all $k \in \mathcal{M}$. If $\frac{H|_{t=0}}{d^\star} < 1$ and $\frac{d_k(\mathbf{n}_{k_1}, \mathbf{n}_{k_2})|_{t=0}}{d^\star} < \frac{1}{M}\left(1 - \frac{H|_{t=0}}{d^\star}\right)$, for all edges k, then the group of unit vectors converges to a configuration where all unit vectors are synchronized.

Proof: Once more, we conclude that $\mathbf{e}$ converges to the null space of $B \otimes \mathbf{I}$ following the same very steps as in the proof of Theorem 4.

Also, since the Lyapunov function is non-increasing, it follows that $V(t) \leq V(0)$. Given the Theorem's conditions it follows that $V(0) < d^\star$. Since $V(t) \leq V(0)$, it follows that $d_k(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}) < d^\star$ for all time $t \geq 0$ and for all edges k. This implies that the angular displacement between any two neighbors is smaller than $\frac{\pi}{2}\frac{1}{N-1}$, i.e., $d_\theta(\mathbf{n}_{k_1}, \mathbf{n}_{k_2}) < f_k^{-1}(d^\star)$. Since, the angular displacement between any two unit vectors $i, j \in \mathcal{N}$ in a connected graph satisfies $d_\theta(\mathbf{n}_i, \mathbf{n}_j) < (N-1)\max_k d_\theta(\mathbf{n}_{k_1}, \mathbf{n}_{k_2})$, it then follows that $d_\theta(\mathbf{n}_i, \mathbf{n}_j) < \frac{\pi}{2}$ for all time $t \geq 0$ and between any two unit vectors. As such, under the Theorem's conditions all unit vectors are contained in a $\frac{\pi}{2}$-cone for all time. Since all agents are contained in an open $\frac{\pi}{2}$-cone, two neighbors will never be diametrically opposed which means control law (8) is well defined for g_k of class $\bar{\mathcal{P}}$. Finally, we invoke Proposition 7, which implies that all unit vectors converge to one another. ■

VII. Constrained Torque

We notice that the control law (7), requires full torque actuation, i.e., it requires torque along $\Pi(\bar{\mathbf{n}}_i)$, but also torque along $\bar{\mathbf{n}}_i$ (since $\boldsymbol{\omega}_i$ can have a component along $\bar{\mathbf{n}}_i$). A natural constraint in a physical system is to require the torque to be orthogonal to $\bar{\mathbf{n}}_i$ (in satellites, for example, thrusters that provide torque along $\bar{\mathbf{n}}_i$ might be unavailable).

A possible approach is to modify the control law into

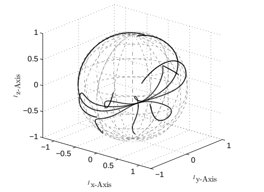
$$\mathbf{T}_i = -\boldsymbol{\sigma}(\Pi(\bar{\mathbf{n}}_i)\boldsymbol{\omega}_i) - \sum_{j \in \mathcal{N}_i} \mathcal{S}(\bar{\mathbf{n}}_i)\mathcal{R}_i^T \frac{\partial d_{\kappa(i,j)}(\mathbf{n}_i, \mathbf{n}_j)}{\partial \mathbf{n}_i}, \quad (13)$$

which results in V in (10) having time derivative $\dot{V} = -\sum_{i=1}^N \boldsymbol{\omega}_i^T \Pi(\bar{\mathbf{n}}_i)\boldsymbol{\sigma}(\boldsymbol{\omega}_i) \leq 0$. Since V will be bounded by its initial condition, all $\boldsymbol{\omega}_i$ are also bounded (V depends on $\|\boldsymbol{\omega}_i\|$ and not on $\|\Pi(\bar{\mathbf{n}}_i)\boldsymbol{\omega}_i\|$), and we can guarantee uniform continuity of $\dot{V}$ and $\dot{\boldsymbol{\omega}}_i$. Our concern is to determine whether $(B \otimes \mathbf{I})\mathbf{e}$ converges to $\mathbf{0}$ and, that is the case, if we can guarantee that all $\mathbf{T}_i$ converge to $\mathbf{0}$. With that in mind, we find that in the set where $\dot{V} = 0$, it holds that $\mathbf{T}_i = (\bar{\mathbf{n}}_i^T\boldsymbol{\omega}_i)^2\mathcal{S}(\bar{\mathbf{n}}_i)J_i\bar{\mathbf{n}}_i$. Since $\bar{\mathbf{n}}_i^T\boldsymbol{\omega}_i$ is not necessarily 0, the only way $\mathbf{T}_i = \mathbf{0}$ is by requiring that $\mathcal{S}(\bar{\mathbf{n}}_i)J_i\bar{\mathbf{n}}_i = \mathbf{0}$; this, in turn, can only be the case when $\bar{\mathbf{n}}_i$ is an eigenvector of J_i. As such, if $\bar{\mathbf{n}}_i$ is not an eigenvector of J_i, control law (7) must be applied and torque along $\bar{\mathbf{n}}_i$ is necessary. On the other hand, if $\bar{\mathbf{n}}_i$ is an eigenvector of J_i, control law (13) can be applied and torque along $\bar{\mathbf{n}}_i$ is not necessary. Physically, this means that if we are trying to perform synchronization of principal axes, full torque is not required, while synchronization of other axes does require full torque. Details of the above derivations can be found in [16].

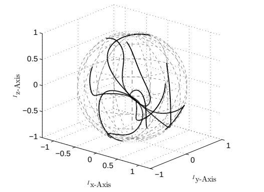
VIII. Simulations

We now present simulations that support some of the results previously presented.

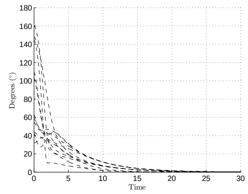
For the simulations, we have a group of ten agents, whose network graph is that presented in Fig. 3. The moments of inertia were generated by adding a random symmetric matrix (between $-\mathbf{I}$ and $\mathbf{I}$) to the identity matrix. For the initial conditions, we have chosen $H|_{t=0} = 0$ and we have randomly generated two sets of 10 rotation matrices. Also,



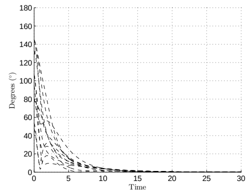
(a) Trajectories of unit vectors in unit sphere #1



(b) Trajectories of unit vectors in unit sphere #2



(c) Error angle between neighbor unit vectors #1



(d) Error angle between neighbor unit vectors #2

Fig. 2. Consensus in network of 10 unit vectors for two sets of differential initial conditions (set #1 and #2)

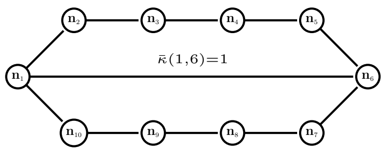


Fig. 3. Graph with 10 agents, where edge 1 is formed by agents 1 and 6

$\bar{\mathbf{n}}_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ for all 10 agents (since these are not necessarily principal axes, we apply control laws (7)).

For edge 1, we have chosen $d(\mathbf{n}_1, \mathbf{n}_2) = 5\tan^2\left(\frac{\arccos(\mathbf{n}_1^T\mathbf{n}_2)}{2}\right)$. For the other edges, we have chosen $d(\mathbf{n}_1, \mathbf{n}_2) = 5(1 - \mathbf{n}_1^T\mathbf{n}_2)$ ($g^\pi = 5$). Notice that we have chosen a distance function (for edge 1) that grows unbounded when two unit vectors are diametrically opposed. As such, it follows from our previous results that agents 1 and 6 will never be diametrically opposed, under the condition that they are not initially diametrically opposed.

We have also chosen $\boldsymbol{\sigma}(\mathbf{x}) = k\frac{\sigma_x \mathbf{x}}{\sqrt{\sigma_x^2 + \mathbf{x}^T\mathbf{x}}}$ with $k = 10$ and $\sigma_x = 1$. For this choice, we find that $\sigma^{\max} = k\sigma_x = 10$. As such, for all agents, except 1 and 6, an upper bound on the norm of their torque is given by $\sigma^{\max} + 2\,g^\pi = 20$ (the factor 2 relates to the fact that all agents, except 1 and 6, have two neighbors). For agents 1 and 6, no upper bound can be found (more precisely, a bound can be found, but it depends on the initial conditions). For these choices, $d^\star \approx 0.019$, which means that if the initial distance between every pair of neighbor agents is smaller than $\frac{d^\star}{M} \approx 0.0017$, then convergence to a synchronized network is guaranteed. This critical value can be made larger by choosing other distance functions, but it comes at the cost of increasing the upper bound on the norm of the torque.

The trajectories of the unit vectors for the two sets of initial conditions are presented in Fig. 2. Notice that despite not satisfying conditions of Theorem 8 (the unit vectors are not always in a $\frac{\pi}{2}$-cone), attitude synchronization is still achieved. This can be verified in Figs. 2(c) and 2(d), which present the angular error between neighbor agents.

IX. Conclusions

In this paper, we proposed a distributed control strategy that guarantees attitude synchronization of unit vectors, representing a specific body direction of a rigid body. The proposed control torque laws depend on distance functions in $\mathcal{S}^2$, and we provide conditions on these distance functions that guarantee that *i)* a synchronized network is locally asymptotically stable in an arbitrary connected undirected graph network; *ii)* a synchronized network can be achieved for almost all initial conditions in a tree graph network. We imposed conditions on the distance functions that guarantee that the proposed control laws can be implemented by each individual rigid body in the absence of a global common orientation frame. Additionally, if the direction to be synchronized is a principal axis of the rigid body, we proposed a control law that does not require full torque actuation. We also studied the equilibria configurations that come with certain type of graph networks. Directions for future work include studying the stability of all equilibria configurations, apart from the synchronized configuration.

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