FEL 3330: Networked and Multi-Agent Control Systems

Lecture 2 compendium: Graphs and Matrices

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Undirected Graphs

Vertices, Edges, Paths

An *undirected graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a nonempty finite set $\mathcal{V}$ of elements called *vertices* (or nodes) and a finite set $\mathcal{E}$ of *edges* (or links), where each element in $\mathcal{E}$ is an unordered pair of two vertices in $\mathcal{V}$.

We often write $\mathcal{V} = \{1, \ldots, n\}$ and an edge between vertices i and j is denoted as $\{i, j\}$. We call two vertices $i, j \in \mathcal{V}$ *adjacent* if $\{i, j\} \in \mathcal{E}$. We call edge $\{i, j\}$ *incident* with its vertices i and j. The *neighborhood* $\mathcal{N}(i) \subseteq \mathcal{V}$ of vertex i is defined as the set of vertices that are adjacent with i, i.e., $\mathcal{N}(i) := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\}$.

A path between two vertices v_1 and v_k in $\mathcal{G}$ is an alternating sequence of distinct vertices

$$v_1 v_2 \ldots v_k$$

such that for any $m = 1, \ldots, k-1$, there is an edge between v_m and v_{m+1}.

We call graph $\mathcal{G}$ *connected* if, for every pair of distinct vertices in $\mathcal{V}$, there is a path between them.

Degree, Adjacency, Incidence, and Laplacian Matrices

The *degree* of node i in graph $\mathcal{G}$, d_i, is the cardinality of $\mathcal{N}(i)$. The *degree matrix* of $\mathcal{G}$, denoted $D(\mathcal{G})$, is the diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$ containing the vertex degrees on the diagonal.

The *adjacency* matrix $A(\mathcal{G})$ is the symmetric $n \times n$ matrix indicating the adjacency relationships in $\mathcal{G}$, in that

$$[A(\mathcal{G})]_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The *Laplacian matrix* of $\mathcal{G}$, denoted $L(\mathcal{G})$, is defined by

$$L(\mathcal{G}) := D(\mathcal{G}) - A(\mathcal{G}) \tag{2}$$

Let m be the number of edges, i.e., the cardinality of $\mathcal{E}$. We now arbitrarily label the edges in $\mathcal{E}$ from 1 to m, and the j'th edge will be denoted as e_j. The *incidence matrix* of $\mathcal{G}$, denoted

$B(\mathcal{G})$, is the symmetric $n \times m$ matrix indicating the incidence relationships in $\mathcal{G}$, in that

$$[B(\mathcal{G})]_{ij} = \begin{cases} 1, & \text{if edge } e_j \text{ is incident with vertex } i, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

We have the following relation.

Proposition 1 $B(\mathcal{G})B(\mathcal{G})^T = D(\mathcal{G}) + A(\mathcal{G})$.

Proof. Plain calculation. □

Incidence Matrix with Orientation

We now assign an *orientation* to $\mathcal{G}$ in the sense that for every edge $\{i, j\} \in \mathcal{V}$, we select one node as *head* and the other *tail*. Intuitively every edge in $\mathcal{E}$ is then equipped with a *direction* starting from the tail pointing to the head. Let $\mathcal{G}^o$ be one of the oriented graph of $\mathcal{G}$. The *oriented incidence matrix* of $\mathcal{G}$, denoted $B_*(\mathcal{G}^o)$, is the symmetric $n \times m$ matrix indicating the oriented incidence relationships in $\mathcal{G}^o$, in that

$$[B_*(\mathcal{G}^o)]_{ij} = \begin{cases} -1, & \text{if } i \text{ is the tail of the edge } e_j, \\ 1, & \text{if } i \text{ is the head of the edge } e_j, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

We have the following relation.

Proposition 2 $B_*(\mathcal{G}^o)B_*(\mathcal{G}^o)^T = D(\mathcal{G}) - A(\mathcal{G}) = L(\mathcal{G})$.

Proof. Again, plain calculation. □

The Laplacian

The Laplacian $L(\mathcal{G})$ is of fundamental importance for characterizing the graph $\mathcal{G}$. We summarize some important properties of $L(\mathcal{G})$:

- Denote $\mathbf{1} = [1, \dots, 1]^T$. Then $L(\mathcal{G})\mathbf{1} = 0$. As a result, zero is an eigenvalue of $L(\mathcal{G})$ with $\mathbf{1}$ a corresponding eigenvector.

- $L(\mathcal{G})$ is symmetric and positive semi-definite. Hence we can sort its eigenvalues by

$$0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \ldots \leq \lambda_n(\mathcal{G}).$$

- Denote $x = (x_1, \ldots, x_n)^T$ as a vector in $\Re^n$. Then

$$x^T L(\mathcal{G})x = x^T B_*(\mathcal{G}^o) B_*(\mathcal{G}^o)^T x = \sum_{\{i,j\}\in\mathcal{E}} (x_i - x_j)^2. \tag{5}$$

- $\lambda_2(\mathcal{G}) = \min_{x\perp\mathbf{1},||x||=1} x^T L(\mathcal{G})x$; $\lambda_n(\mathcal{G}) = \max_{||x||=1} x^T L(\mathcal{G})x$.

We end the discussion on undirected graphs with the following theorem.

Theorem 1 *The graph $\mathcal{G}$ is connected if and only if $\lambda_2(\mathcal{G}) > 0$.*

Proof: It is crucial to notice that $\mathcal{G}$ is connected if and only if the null space of $B_*(\mathcal{G}^o)^T$ has dimension one.

If $\lambda_2(\mathcal{G}) > 0$, then the null space of $L(\mathcal{G})$ has dimension one. This in turn implies that the null space of $B_*(\mathcal{G}^o)^T$ has dimension one based on Proposition 2. The graph $\mathcal{G}$ is thus connected. This proves the sufficiency.

If $\lambda_2(\mathcal{G}) = 0$, then the null space of $L(\mathcal{G})$ has dimension at least two. This in turn implies that the null space of $B_*(\mathcal{G}^o)^T$ has dimension at least two based on Proposition 2. The graph $\mathcal{G}$ is thus not connected. This proves the necessity. $\square$

Directed Graphs

Arcs, Paths

A *directed graph* (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a nonempty finite set $\mathcal{V} = \{1, \dots, n\}$ of *nodes*, and a finite set $\mathcal{E}$ of *arcs*, where each element in $\mathcal{E}$ is an ordered pair of two vertices in $\mathcal{V}$. An element $(i, j) \in \mathcal{E}$ starts from i and points to j. We call i the tail and j the head of in arc (i, j).

A directed path from vertex v_1 to v_k in digraph $\mathcal{G}$ is an alternating sequence of distinct nodes

$$v_1 v_2 \dots v_k$$

such that $(v_m, v_{m+1}) \in \mathcal{E}$ for any $m = 1, \dots, k - 1$.

Connectivity

We call node j *reachable* from node i if there is a directed path from i to j in digraph $\mathcal{G}$. In particular every node is supposed to be reachable from itself. A node v from which every node in $\mathcal{V}$ is reachable is called a *center node* (root).

A digraph $\mathcal{G}$ is *strongly connected* if every two nodes are mutually reachable; $\mathcal{G}$ is *quasi-strongly connected* if for every two nodes i and j, there is a node u from which both i and j are reachable; $\mathcal{G}$ is *weakly connected* if we can get a connected undirected graph by removing all the directions of the arcs in $\mathcal{E}$.

For the connectivity of digraphs, the following theorem holds [2].

Theorem 2 *A digraph $\mathcal{G}$ is quasi-strongly connected if and only if $\mathcal{G}$ has at least one center node.*

Proof. The sufficiency claim is trivial. Now suppose $\mathcal{G}$ is quasi-strongly connected. Take two different nodes i and j. There is a node $v(i, j)$ (as a function of i and j) from which both i and j are reachable. This node $v(i, j)$ might be within $\{i, j\}$, or not. We anyhow denote $V_1 = \{i, j, v(i, j)\}$. Then take a node outside V_1, say k. There must be a node $u(V_1, k)$ from which every node in $V_1 \cup \{k\}$ is reachable. We can repeat this argument until every node in the graph has been visited. The last node we find is clearly a center node of $\mathcal{G}$. $\square$

The Digraph Lapalacian

In a digraph $\mathcal{G}$, the *in-neighborhood* $\mathcal{N}^+(i) \subseteq \mathcal{V}$ of node i is defined as $\mathcal{N}^+(i) := \{j \in \mathcal{V} : \{j, i\} \in \mathcal{E}\}$. The *out-neighborhood* $\mathcal{N}^-(i) \subseteq \mathcal{V}$ of node i is defined as $\mathcal{N}^-(i) := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\}$. The *in-degree* of node i in digraph $\mathcal{G}$, d_i^+, is the cardinality of $\mathcal{N}^+(i)$. The *out-degree* of node i in digraph $\mathcal{G}$, d_i^-, is the cardinality of $\mathcal{N}^-(i)$.

The *in-degree matrix* of digraph $\mathcal{G}$, denoted $D^+(\mathcal{G})$, is the diagonal matrix $\text{diag}(d_1^+, \ldots, d_n^+)$. The *adjacency* matrix $A(\mathcal{G})$ of digraph $\mathcal{G}$ is the $n \times n$ matrix given by

$$[A(\mathcal{G})]_{ij} = \begin{cases} 1, & \text{if } (j, i) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

The *Laplacian matrix* of digraph $\mathcal{G}$, denoted $L(\mathcal{G})$, is defined by

$$L(\mathcal{G}) := D^+(\mathcal{G}) - A(\mathcal{G}) \tag{7}$$

Unlike the case for undirected graphs, $L(\mathcal{G})$ is no longer symmetric. However, it is still obvious to verify that $L(\mathcal{G})\mathbf{1} = 0$. This means, for directed graphs, zero continues to be an eigenvalue of $L(\mathcal{G})$ and $\mathbf{1}$ is a corresponding eigenvector. It is also worth mentioning that the eigenvalues of $L(\mathcal{G})$ always lie in the closed Left Half-Plane as a direct conclusion from the Geršhgorin circle theorem.

The following result holds. The proof is omitted since it requires considerably more preliminary knowledge. We refer to [3] for a complete proof.

Theorem 3 *Let $L(\mathcal{G})$ be the Laplacian of a digraph $\mathcal{G}$. Then* $\text{rank}L(\mathcal{G}) = n - 1$ *if and only if $\mathcal{G}$ is quasi-strongly connected.*

Bibliography

[1] C. Godsil and G. Royle. Algebraic Graph Theory. Springer Graduate Texts in Mathematics, 2001.

[2] C. Berge and A. Ghouila-Houri. Programming, Games and Transportation Networks. John Wiley and Sons, 1965.

[3] M. Mesbahi and M. Egerstedt. Graph Theoretic Methods in Multiagent Networks. Princeton, 2010.