FEL 3330: Networked and Multi-Agent Control Systems
Lecture 1 compendium: Essentials of Algebraic Graph Theory

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An undirected graph $G = \{V, E\}$ consists of a set of vertices $V = \{1, ..., N\}$ and a set of edges, $E = \{(i, j) \in V \times V\}$ containing pairs of vertices.

For an undirected graph $G = \{V, E\}$ with N vertices $V = \{1, \ldots, N\}$ and edges $E \subset V \times V$, the *adjacency matrix* $A = A(G) = (a_{ij})$ is the $N \times N$ matrix given by $a_{ij} = 1$, if $(i, j) \in E$, and $a_{ij} = 0$, otherwise. If $(i, j) \in E$, then i, j are *adjacent.* A *path* of length r from i to j is a sequence of $r + 1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. For $i = j$, this path is a *cycle.* If there is a path between any two vertices of G, then G is *connected.* A connected graph is a *tree* if it contains no cycles. The *degree* d_i of vertex i is given by $d_i = \sum_j a_{ij}$. Let $\Delta = \mathrm{diag}(d_1, \ldots, d_N)$. The *Laplacian* of G is the symmetric positive semidefinite matrix

$$L = \Delta - A$$

For a connected graph, L has a simple zero eigenvalue with the corresponding eigenvector $\mathbf{1} = [1, \ldots, 1]^T$. This will be formally stated in Theorem 2 below. We denote by $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_N(G)$ the eigenvalues of L.

Two important relations resulting from the symmetry of L and the variational characterization of the eigenvalues of symmetric matrices are as follows:

$$\lambda_2(G) = \min_{x \perp \mathbf{1}, ||x||=1} x^T L x$$

and

$$\lambda_N(G) = \max_{||x||=1} x^T L x$$

An orientation on G is the assignment of a direction to each edge. The incidence matrix $B = B(G) = (b_{ij})$ of an oriented graph is the $\{0, \pm 1\}$-matrix with rows and columns indexed by the vertices and edges of G, respectively, such that $b_{ij} = 1$ if the vertex i is the head of the edge j, $b_{ij} = -1$ if the vertex i is the tail of the edge j, and $b_{ij} = 0$ otherwise. It can be shown that $L = BB^T$, and this is independent oft he choice of orientation.

If G contains cycles, the edges of each cycle have a direction, where each edge is directed towards its successor according to the cyclic order. A cycle C is represented by a vector v_C with $M = |E|$ elements. For each edge, the corresponding element of v_C is equal to 1 if the direction of the edge with respect to C coincides with the orientation assigned to the graph for defining B, and -1, if the direction with respect to C is opposite to the orientation. The elements corresponding to edges not in C are zero. The *cycle space* of G is the subspace spanned by vectors representing cycles in G [2].

Let $x = [x_1, \ldots, x_N]^T$, where x_i is a real scalar variable assigned to vertex i of G. Denote by $\bar{x}$ the M-dimensional stack vector of relative differences of pairs of agents that form an edge in G, where $M = |E|$ is the number of edges, in agreement with a defined orientation. In particular, denoting by $e_i = (h_i, t_i) \in E$, $i = 1, \ldots, M$, the edges of G, where h_i,t_i the head and tail of e_i respectively, we denote $\bar{x}_{e_i} = x_{h_i} - x_{t_i}$. The vector $\bar{x}$ is given by $\bar{x} = [\bar{x}_{e_1}, \ldots, \bar{x}_{e_M}]^T$. It is easy to verify that $Lx = B\bar{x}$ and $\bar{x} = B^T x$. For $\bar{x} = 0$ we have that $Lx = 0$.

Lemma 1 *If G is a tree, then $B^T B$ is positive definite.*

Proof: For any $y \in \mathbb{R}^M$, we have $y^T B^T By = |By|^2$ and hence $y^T B^T By > 0$ if and only if $By \neq 0$, i.e., the matrix B has empty null space. For a connected graph, the cycle space of the graph coincides with the null space of B (Lemma 3.2 in [2]). Thus, for G with no cycles, zero is not an eigenvalue of B. This implies that $B^T B$ is positive definite. $\diamondsuit$

The following theorem also holds:

Theorem 2 *The graph G is connected if and only if $\lambda_2(G) > 0$.*

Proof: (Sketch, full proof at [1]). We can show that B^T and L have the same null space, so that it suffices to show that the null space of B^T has dimension one, or that its rank is $n - 1$ when G is connected. Suppose that z is a vector such that $z^T B = 0$. This then implies that for $(i, j) \in E$, then $z_i - z_j = 0$. Since G is connected, this means $z \in \mathbf{span}\{\mathbf{1}\}$. Thus the rank of the null space of B^T and thus, L, is one, which implies that the multiplicity of $\lambda_1(G) = 0$ is one. $\diamondsuit$

Bibliography

[1] C. Godsil and G. Royle. *Algebraic Graph Theory.* Springer Graduate Texts in Mathematics # 207, 2001.

[2] S. Guattery and G.L. Miller. Graph embeddings and laplacian eigenvalues. *SIAM Journ. Matrix Anal. Appl.*, 21(3):703–723, 2000.