Decentralized Abstractions for Feedback Interconnected Multi-Agent Systems

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***Abstract*— The purpose of this paper is to define abstractions for multi-agent systems under coupled constraints. In the proposed decentralized framework, we specify a finite or countable transition system for each agent which only takes into account the discrete positions of its neighbors. The dynamics of the considered systems consist of two components. An appropriate feedback law which guarantees that certain performance requirements (e.g., connectivity) are preserved and induces the coupled constraints, and additional free inputs which are exploited for the accomplishment of high level tasks. In this work we provide sufficient conditions on the space and time discretization for the abstraction of the system's behaviour which ensure that we can extract a well posed and hence meaningful transition system.**

I. Introduction

Task planning under temporal logic specifications constitutes a highly active area of research which lies in the interface between computer science and modern control theory. One main challenge in this new interdisciplinary direction is the problem of defining appropriate abstractions for continuous time multi-agent control systems and hence enabling the analysis and control of large scale systems or the achievement of high level plans. Robot motion planning and control constitutes a central field where this line of work is applied. In particular the use of a suitable discrete system's model allows the specification of high level plans, which under an appropriate equivalence notion between the continuous system and its discrete analog, can be converted to low level primitives such as feedback controllers, that are able to implement the high level tasks. Such tasks in the case of multiple mobile robots in an industrial workspace could include for example the following scenario. Robot 1 should periodically go from region A to region B, while avoiding C, and after collecting an item of type X from robot 2 at location D, store it at location E.

In order to accomplish high level plans, it is required to specify a finite abstraction of the original system, namely a system that preserves some properties of interest of the initial system, while ignoring detail. Results in this direction for the nonlinear centralized case have been obtained in the papers [10], [16] where the notions of approximate bisimulation and simulation are exploited for certain classes of nonlinear systems under appropriate stability assumptions. Another tool towards this direction is the hybridization approach [1], where the behaviour of a nonlinear system is abstracted by means of a piecewise affine hybrid system on simplices. Motion planing techniques for the latter case have been developed in [4]. Recent extensions to the case of discrete time networked systems that are described by coupled difference equations, include [14] and [11], where finite abstractions are provided for stabilizable linear systems and incrementally input-to-state stable nonlinear systems, respectively.

In this framework, we focus on multi-agent systems and assume that the agents' dynamics consist of feedback interconnection terms, which ensure that certain system properties as for instance connectivity or (and) invariance are preserved, and free input terms, which provide the ability for motion planning under the coupled constraints. To the best of our knowledge, this is the first attempt to provide decentralized abstractions for continuous time multi-agent systems in the presence of coupled constraints that are induced through their feedback interconnection. In this paper, admissible space-time discretizations which are used in order to capture reachability properties of the original system are quantified and sufficient conditions which establish that the system's abstraction is well posed are provided. The latter ensure that for each agent, the finite transition system which serves as an abstract model of the agent's behaviour has at least one outgoing transition for each discrete state.

The rest of the paper is organized as follows. Basic notation and preliminaries are introduced in Section II. In Section III, we define well posed abstractions for single integrator multi-agent systems by means of hybrid feedback controllers and prove that the latter provide solutions consistent with our design requirement on the systems' free inputs. In Section IV, space-time discretizations that guarantee well posed abstractions are quantified. We conclude and indicate directions of further research in Section V.

II. Preliminaries and Notation

We use the notation $|x|$ for the Euclidean norm of a vector $x \in \mathbb{R}^n$. For a subset S of $\mathbb{R}^n$, we denote by $\mathrm{cl}(S)$, $\mathrm{int}(S)$ and ∂S its closure, interior and boundary, respectively, where $\partial S := \mathrm{cl}(S) \setminus \mathrm{int}(S)$. Given $R > 0$ and $y \in \mathbb{R}^n$, we denote by $B(R)$ the closed ball with center $0 \in \mathbb{R}^n$ and radius R, namely $B(R) := \{x \in \mathbb{R}^n : |x| \leq R\}$ and $B_y(R) := \{x \in \mathbb{R}^n : |x - y| \leq R\}$. Given two sets $A, B \in \mathbb{R}^n$ their Minkowski sum is defined as $A + B := \{x + y \in \mathbb{R}^n : x \in A, y \in B\}$.

Consider a multi-agent system with N agents. For each agent $i \in \{1, \ldots, N\}$ we use the notation $\mathcal{N}_i$ for the

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set of its neighbors and $|\mathcal{N}_i|$ for its cardinality. We also consider an ordering of the agent's neighbors which we denote by $j_1,\ldots,j_{|\mathcal{N}_i|}$. Given an index set $\mathcal{I}$ and an agent $i\in\{1,\ldots,N\}$ with neighbors $j_1,\ldots,j_{|\mathcal{N}_i|}\in\{1,\ldots,N\}$, we define the mapping $\mathrm{pr}_i:\mathcal{I}^N\to\mathcal{I}^{|\mathcal{N}_i|+1}$ which assigns to each N-tuple $(l_1,\ldots,l_N)\in\mathcal{I}^N$ the $|\mathcal{N}_i|+1$-tuple $(l_i,l_{j_1},\ldots,l_{j_{|\mathcal{N}_i|}})\in\mathcal{I}^{|\mathcal{N}_i|+1}$.

We proceed by providing a formal definition for the notion of a transition system (see for instance [2], [9], [10]).

Definition 2.1: A transition system is a quintuple $TS:=(Q,L,\longrightarrow,O,H)$, where: Q is a set of states; L is a set of actions; $\longrightarrow$ is a transition relation with $\longrightarrow\subset Q\times L\times Q$; O is an output set and H is an output function from Q to O. The transition system is said to be finite, if Q and L are finite sets. We also use the (standard) notation $q\overset{l}{\longrightarrow}q'$ to denote an element $(q,l,q')\in\longrightarrow$. For every $q\in Q$ and $l\in L$ we use the notation $\mathrm{Post}(q;l):=\{q'\in Q:(q,l,q')\in\longrightarrow\}$.

III. Abstractions for Multi-Agent Systems

We focus on multi-agent systems with single integrator dynamics

$$\dot{x}_i=u_i,x_i\in\mathbb{R}^n,i=1,\ldots,N \tag{1}$$

and consider as inputs decentralized control laws of the form

$$u_i=f_i(x_i,x_{j_1},\ldots,x_{j_{|\mathcal{N}_i|}})+v_i,i=1,\ldots,N \tag{2}$$

consisting of two terms, the feedback term $f_i(\cdot)$ which depends on the states of i and its neighbors, and the free input v_i. We assume that for each $i=1,\ldots,N$ it holds that $x_i\in D$ where D is a domain of $\mathbb{R}^n$ and that each $f_i(\cdot)$ is locally Lipschitz.

In order to justify our subsequent analysis, we assume that the f_i's are globally bounded and that the maximum magnitude of the feedback terms is higher than that of the free inputs, since we are primarily interested in maintaining the property that the feedback is designed for and, secondarily, in exploiting the free inputs in order to accomplish high level tasks. In what follows, we consider a cell decomposition of the state space D (which can be regarded as a partition of D) and a time discretization step $\delta t>0$. In particular, we adopt a modification of the corresponding definition from [6, p. 129-called cell covering].

Definition 3.1: Let D be a domain of $\mathbb{R}^n$. A cell decomposition $\mathcal{S}=\{S_l\}_{l\in\mathcal{I}}$ of D, where $\mathcal{I}$ is a finite or coutable index set, is a finite or countable family of uniformly bounded sets S_l, $l\in\mathcal{I}$ whose interior is a domain, such that $\mathrm{int}(S_l)\cap\mathrm{int}(S_{\hat{l}})=\emptyset$ for all $l\neq\hat{l}$ and $\cup_{l\in\mathcal{I}}S_l=D$. ◁

Our ultimate goal is to define finite abstractions for closed loop multi-agent systems of the form (1)-(2) which evolve inside a bounded domain and satisfy the following invariance assumption.

(IA) For every initial condition $x(0)\in D^N$ of system (1)-(2) and selection of the v_i's from a bounded subset of $L^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^n)$, the unique solution of (1)-(2) is defined for all $t\geq 0$ and remains in D^N (for all $t\geq 0$).

A motivating example for this framework has been studied in our companion work [3] where both network connectivity and invariance of the system's solution are established for the single integrator model evolving inside a bounded domain. Furthermore, robustness of these properties with respect to free inputs is guaranteed. A finite cell decomposition in that case can lead to a finite transition system which captures the properties of interest of the multi-agent system and hence enables the investigation for computable solutions with respect to high level plan specifications.

A basic feature that we want to satisfy through our space and time discretization is the possibility to maintain some of the reachability properties of the original system, when we consider the finite transition system that results from the cell decomposition and the time discretization. Informally, we would like to consider for each agent i its individual transition system whose states are all the possible modes of the cell decomposition, namely the cells of the state partition and whose actions are all the possible cells of agent i's neighbors. Then, a discrete transition from an initial cell to a final one should be feasible for i, *if for all states in the initial cell there exists a free input, such that the trajectory of i will reach the final cell at time δt, for all possible initial states of its neighbors in their cells and their corresponding free inputs*. High level planning requires each individual transition system to be well posed-meaningful, which implies that each agent can transit from each initial cell to (at least) one final cell.

One main challenge in the attempt to provide meaningful decentralized abstractions even in this fully actuated with respect to the free inputs case is the interconnection between the agents through the $f_i(\cdot)$ terms. The latter leads us to reformulate our informal consideration above and motivates the design of appropriate hybrid feedback laws in the place of the v_i's which will guarantee the desired well posed transitions. Before proceeding to the necessary definitions related to our problem formulation, we provide some bounds on the dynamics of the multi-agent system. In order to simplify the subsequent analysis, which we aim to appropriately modify in order to include domains satisfying (IA) and hence extract finite transition systems, we assume for (1)-(2) that $D=\mathbb{R}^n$. We also assume that the feedback terms $f_i(\cdot)$ are globally bounded, namely, there exists a constant $M>0$ such that for all $(x_i,x_{j_1},\ldots,x_{j_{|\mathcal{N}_i|}})\in\mathbb{R}^{(|\mathcal{N}_i|+1)n}$ it holds

$$|f_i(x_i,x_{j_1},\ldots,x_{j_{|\mathcal{N}_i|}})|\leq M \tag{3}$$

Furthermore, we require that the free inputs v_i satisfy the bound

$$|v_i(t)|\leq v_{\max},\forall t\geq 0,i=1,\ldots,N \tag{4}$$

Given the time step δt, and the bounds M and $v_{\max}$ on the feedback and input terms, we introduce the following lengthscale

$$R_{\max}:=\delta t(M+v_{\max}) \tag{5}$$

with M and $v_{\max}$ as given in (3) and (4), respectively. It follows from (1), (2), (3), (4) and (5) that $R_{\max}$ is the maximum distance an agent can travel within time δt.

Given a cell decomposition $\mathcal{S}:=\{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$, we use the notation $\tilde{l}_i=(l_i,l_i^1,\ldots,l_i^{|\mathcal{N}_i|})\in\mathcal{I}^{|\mathcal{N}_i|+1}$ to denote the indices of the cells where agent i and its neighbors belong at a certain time instant (e.g. at $t = 0$) and call it the (initial) cell configuration of i. Similarly, we use the notation $\bar{l} = (\bar{l}_1, \dots, \bar{l}_N) \in \mathcal{I}^N$ to specify the indices of the cells where all the N agents belong at a given time instant and call it the cell configuration (of all agents). Thus, given a cell configuration $\bar{l}$ we can determine the cell configuration $\tilde{l}_i$ of agent i through the mapping $\mathrm{pr}_i : \mathcal{I}^N \to \mathcal{I}^{|\mathcal{N}_i|+1}$, namely $\tilde{l}_i = \mathrm{pr}_i(\bar{l})$ (see Section II for the definition of $\mathrm{pr}_i(\cdot)$). In this paper, we are primarily interested in the evolution of the system on the time interval $[0, \delta t]$, since we focus on the transitions from initial states at $t = 0$ to final states at $t = \delta t$. Thus, we will also use the term final cell configuration when referring to the time instant δt.

Before defining the notion of a well posed space time discretization we provide a class of hybrid feedback laws, parameterized by the agents' initial conditions, which we assign to the free inputs v_i in order to obtain meaningful discrete transitions.

Definition 3.2: Given a space-time discretization $\mathcal{S} - \delta t$ ($\mathcal{S} := \{S_l\}_{l \in \mathcal{I}}$), an agent $i \in \{1, \dots, N\}$ and an initial cell configuration $\tilde{l}_i = (l_i, l_i^1, \dots, l_i^{|\mathcal{N}_i|}) \in \mathcal{I}^{|\mathcal{N}_i|+1}$ of i, we say that the mapping $\mathbb{R}_{\geq 0} \times \mathbb{R}^{(|\mathcal{N}_i|+1)n} \times \mathbb{R}^n \ni (t, x_i, x_{j_1}, \dots, x_{j_{|\mathcal{N}_i|}}; x_{i0}) \to k_{i,\tilde{l}_i}(t, x_i, x_{j_1}, \dots, x_{j_{|\mathcal{N}_i|}}; x_{i0}) \in \mathbb{R}^n$ satisfies property **(P)**, if the following hold.

(P1) For each $x_{i0} \in \mathbb{R}^n$ the mapping $k_{i,\tilde{l}_i}(\cdot; x_{i0}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^{(|\mathcal{N}_i|+1)n} \to \mathbb{R}^n$ is locally Lipschitz continuous.

(P2) It holds $|k_{i,\tilde{l}_i}(t, x_i, x_{j_1}, \dots, x_{j_{|\mathcal{N}_i|}}; x_{i0})| \leq v_{\max}, \forall t \in [0, \delta t], x_i \in S_{l_i} + B(R_{\max}), x_{j_\kappa} \in S_{l_i^\kappa} + B(R_{\max}), \kappa = 1, \dots, |\mathcal{N}_i|, x_{i0} \in S_{l_i}$, with $v_{\max}$ as given in (4) and $R_{\max}$ as in (5). $\triangleleft$

We next provide the definition of a well posed space-time discretization, in accordance to our previous discussions.

Definition 3.3: Consider a cell decomposition $\mathcal{S} = \{S_l\}_{l \in \mathcal{I}}$ of $\mathbb{R}^n$ and a time step δt.

(a) Given an agent $i \in \{1, \dots, N\}$, an initial cell configuration $\tilde{l}_i = (l_i, l_i^1, \dots, l_i^{|\mathcal{N}_i|}) \in \mathcal{I}^{|\mathcal{N}_i|+1}$ of i and a cell index $l_i' \in \mathcal{I}$ we say that the transition $l_i \xrightarrow{\tilde{l}_i} l_i'$ is well posed with respect to the space-time discretization $\mathcal{S} - \delta t$ if there exists a feedback law

$$v_i = k_{i,\tilde{l}_i}(t, x_i, x_{j_1}, \dots, x_{j_{|\mathcal{N}_i|}}; x_{i0}) \tag{6}$$

parameterized by $x_{i0} \in \mathbb{R}^n$ (the initial condition of i) and satisfying property (P), such that condition (C) below is fulfilled.

(C) For each initial cell configuration $\bar{l} = (\bar{l}_1, \dots, \bar{l}_N) \in \mathcal{I}^N$ with $\mathrm{pr}_i(\bar{l}) = \tilde{l}_i$, for all $\hat{i} \in \{1, \dots, N\} \setminus \{i\}$ and feedback laws

$$v_{\hat{i}} = k_{\hat{i},\tilde{l}_{\hat{i}}}(t, x_{\hat{i}}, x_{\hat{j}_1}, \dots, x_{\hat{j}_{|\mathcal{N}_{\hat{i}}|}}; x_{\hat{i}0}) \tag{7}$$

parameterized by $x_{\hat{i}0} \in \mathbb{R}^n$ (the initial condition of $\hat{i}$) and satisfying property (P) (with $\tilde{l}_{\hat{i}} = \mathrm{pr}_{\hat{i}}(\bar{l})$), and for all initial conditions $x(0) \in \mathbb{R}^n$ with $x_\kappa(0) = x_{\kappa 0} \in S_{\bar{l}_\kappa}$, $\kappa = 1, \dots, N$, the closed loop system (1)-(2)-(6)-(7) (with $v_\kappa = k_{\kappa,\tilde{l}_\kappa}$, $\kappa = 1, \dots, N$) has a unique solution which is defined on $[0, \delta t]$ and satisfies $x_i(\delta t, x(0)) \in S_{l_i'}$.

(b) We say that the space-time discretization $\mathcal{S} - \delta t$ is well posed if for each agent $i \in \{1, \dots, N\}$ and each cell configuration $\tilde{l}_i = (l_i, l_i^1, \dots, l_i^{|\mathcal{N}_i|}) \in \mathcal{I}^{|\mathcal{N}_i|+1}$ of i, there exists a cell index $l_i' \in \mathcal{I}$ such that the transition $l_i \xrightarrow{\tilde{l}_i} l_i'$ is well posed with respect to $\mathcal{S} - \delta t$. $\triangleleft$

Given a space-time discretization $\mathcal{S} - \delta t$ and based on Definition 3.3, we now provide an exact description of the discrete transition system which serves as an abstract model for the behaviour of each agent. We do not focus on the output set and map of the transition system and just provide the definition of its state set, action set and transition relation. In particular, for each agent i, we define the discrete transition system $TS_i := (Q, L_i, \longrightarrow_i)$ with state set Q the indices $\mathcal{I}$ of the cell decomposition, actions all possible cell indices of i and its neighbors, namely $L_i := \mathcal{I}^{|\mathcal{N}_i|+1}$ (the set of all possible cell configurations of i) and transition relation $\longrightarrow_i \subset Q \times L_i \times Q$ defined as follows. For any $\hat{l}_i, \hat{l}_i' \in Q$ and $\tilde{l}_i = (l_i, l_i^1, \dots, l_i^{|\mathcal{N}_i|}) \in \mathcal{I}^{|\mathcal{N}_i|+1}$: $\hat{l}_i \xrightarrow{\tilde{l}_i}_i \hat{l}_i'$ iff $\hat{l}_i = l_i$ and $l_i \xrightarrow{\tilde{l}_i} \hat{l}_i'$ is well posed. We have preferred to use the term actions instead of labels for the elements of the set L_i, because the cell configuration of i indicates how the feedback term $f_i(\cdot)$ acts on and affects the possible transitions of i.

According to Definition 3.3, a well posed space-time discretization requires the existence of a well posed transition for each agent i and the latter reduces to the selection of an appropriate feedback controller for i, which also satisfies Property (P), and the requirement that the selected feedback controllers of the other agents also satisfy (P). Yet, it is not completely evident, that given an initial cell configuration and a well posed transition for each agent, it is possible to choose a distributed feedback law for each agent, so that the resulting closed loop system will guarantee all these well posed transitions (for all possible initial conditions in the cell configuration). The following proposition clarifies this point.

Proposition 3.4: Consider system (1)-(2), let $\bar{l} = (\bar{l}_1, \dots, \bar{l}_N) \in \mathcal{I}^N$ be an initial cell configuration and assume that the space-time discretization $S - \delta t$ is well posed, which implies that for all $i = 1, \dots, N$ it holds that $\mathrm{Post}_i(\bar{l}_i; \mathrm{pr}_i(\bar{l})) \neq \emptyset$ ($\mathrm{Post}_i(\cdot)$ refers to the transition system TS_i of each agent-see also Section II). Then, for every final cell configuration $\bar{l}' = (\bar{l}_1', \dots, \bar{l}_N') \in \mathrm{Post}_1(\bar{l}_1; \mathrm{pr}_1(\bar{l})) \times \cdots \times \mathrm{Post}_N(\bar{l}_N; \mathrm{pr}_N(\bar{l}))$ there exist feedback laws

$$v_i = k_{i,\mathrm{pr}_i(\bar{l})}(t, x_i, x_{j_1}, \dots, x_{j_{|\mathcal{N}_i|}}; x_{i0}), i = 1, \dots, N \tag{8}$$

satisfying property (P) and such that for all initial conditions $x(0) \in \mathbb{R}^{Nn}$ with $x_i(0) = x_{i0} \in S_{\bar{l}_i}$, $i = 1, \dots, N$ the solution of the closed loop system (1)-(2)-(8) (with $v_i = k_{i,\mathrm{pr}_i(\bar{l})}$, $i = 1, \dots, N$) is defined on $[0, \delta t]$ and satisfies

$$x_i(\delta t, x(0)) \in S_{\bar{l}_i'}, \forall i = 1, \dots, N \tag{9}$$

Proof: Indeed, consider a final cell configuration $\bar{l}' = (\bar{l}_1', \dots, \bar{l}_N')$ as in the statement of the proposition and select for each agent $i \in \{1,\dots,N\}$ a control law $k_{i,\mathrm{pr}_i(\bar{l})}(\cdot)$ which ensures that $\bar{l}_i \overset{\mathrm{pr}_i(\bar{l})}{\longrightarrow} \bar{l}_i'$ is well posed. It follows from Definition 3.3(a) that all the feedback laws $k_{i,\mathrm{pr}_i(\bar{l})}(\cdot)$, $i=1,\dots,N$ satisfy Property (P) and hence, from Condition (C), that for each initial condition as in the statement of the proposition, the solution of the closed loop system is defined on $[0,\delta t]$ and satifies (9). ■

The following proposition guarantees that due to Property (P), the selection of the controllers in Definition 3.3 provides well defined solutions for the closed loop system on $[0,\delta t]$ and hence, that the requirement for a unique solution in Condition (C) of Definition 3.3 is redundant. We exploit this result in Proposition 4.1 where we derive sufficient conditions for well posed space-time discretizations. Furthermore, Proposition 3.5 guarantees that the magnitude of the hybrid feedback laws does not exceed the maximum allowed magnitude of the free inputs $v_{\max}$ on $[0,\delta t]$ and hence establishes consistency with our initial design requirement.

Proposition 3.5: Consider the space-time discretization $\mathcal{S}-\delta t$ corresponding to the cell decomposition $\mathcal{S}$ of $\mathbb{R}^n$ and the time step δt. Let $\bar{l}=(\bar{l}_1,\dots,\bar{l}_N)\in\mathcal{I}^N$ be an initial cell configuration and consider the feedback laws

$$v_i = k_{i,\mathrm{pr}_i(\bar{l})}(t,x_i,x_{j_1},\dots,x_{j_{|\mathcal{N}_i|}};x_{i0}), i=1,\dots,N \quad (10)$$

assigned to the agents that satisfy Property (P). Then for all initial conditions $x(0)\in\mathbb{R}^{Nn}$ with $x_i(0)=x_{i0}\in S_{\bar{l}_i}$, $i=1,\dots,N$ the solution of the closed loop system (1)-(2)-(10) (with $v_i=k_{i,\mathrm{pr}_i(\bar{l})}$, $i=1,\dots,N$) is defined on $[0,\delta t]$ and satisfies

$$|k_{i,\mathrm{pr}_i(\bar{l})}(t,x_i(t),x_{j_1}(t),\dots,x_{j_{|\mathcal{N}_i|}}(t);x_{i0})|\le v_{\max}, \quad (11)$$

for all $t\in[0,\delta t]$ and $i=1,\dots,N$, which provides the desired consistency with our design requirement (4) on the v_i's.

Proof: Let $x(0)\in\mathbb{R}^{Nn}$ with $x_i(0)\in S_{\bar{l}_i}$, $i=1,\dots,N$ be the initial condition of the closed loop system. Then it follows from the local Lipschitz property for the functions $f_i(\cdot)$ and the corresponding property for the mappings $k_{i,\mathrm{pr}_i(\bar{l})}(\cdot;x_{i0})$ provided by (P1), that there exists a unique solution $x(\cdot)=x(\cdot,x(0))$ to the initial value problem defined on the right maximal interval of existence $[0,T_{\max})$. The rest of the proof is based on the claim that each component $x_i(\cdot)$, $i=1,\dots,N$ of the solution satisfies

$$x_i(t)\in S_{\bar{l}_i}+B(R_{\max}),\forall t\in[0,\min\{T_{\max},\delta t\}) \quad (12)$$

with $R_{\max}$ as given in (5). Then it follows that $T_{\max}>\delta t$, because on the contrary (12) would imply that $x(t)$ remains in a compact subset of $\mathbb{R}^{Nn}$ for all $t\in[0,T_{\max})$, with $T_{\max}<\infty$, contradicting maximality of $[0,T_{\max})$. Furthermore, from (12), (P2) and continuity of $x(\cdot)$ we get that (11) is satisfied, which provides the desired result. We proceed by proving (12). Indeed, suppose on the contrary that (12) is violated and hence, that there exists $\hat{i}\in\{1,\dots,N\}$ and a time $\tilde{t}$ with

$$\tilde{t}\in(0,\delta t) \text{ and } x_{\hat{i}}(\tilde{t})\notin S_{\bar{l}_{\hat{i}}}+B(R_{\max}) \quad (13)$$

By exploiting continuity of $x(\cdot)$ we may define $\tau:=\max\{\bar{t}\in[0,\tilde{t}]: x_i(t)\in\mathrm{cl}(S_{\bar{l}_i}+B(R_{\max})),\forall t\in[0,\bar{t}], i=1,\dots,N\}$. Then, it follows from the latter and (13) that there exists $\tilde{i}\in\{1,\dots,N\}$ such that

$$x_{\tilde{i}}(\tau)\in\partial(S_{\bar{l}_{\tilde{i}}}+B(R_{\max})) \text{ and } \tau\le\tilde{t}<\delta t \quad (14)$$

It also follows from the definition of τ that $x_i(t)\in\mathrm{cl}(S_{\bar{l}_i}+B(R_{\max})),\forall t\in[0,\tau], i=1,\dots,N$ and thus from Property (P2) and continuity of $x(\cdot)$ and $k_{\tilde{i},\mathrm{pr}_{\tilde{i}}(\bar{l})}(\cdot;x_{\tilde{i}0})$ that for all $t\in[0,\tau]$ it holds $|k_{\tilde{i},\mathrm{pr}_{\tilde{i}}(\bar{l})}(t,x_{\tilde{i}}(t),x_{\tilde{j}_1}(t),\dots,x_{\tilde{j}_{|\mathcal{N}_{\tilde{i}}|}}(t);x_{\tilde{i}0})|\le v_{\max}$. Hence, from the latter, (1), (2), (5), (10) and the inequality in (14) we get that

$$\begin{aligned}|x_{\tilde{i}}(\tau)-x_{\tilde{i}0}|\le&\int_0^\tau[|f_{\tilde{i}}(x_{\tilde{i}}(s),x_{\tilde{j}_1}(s),\dots,x_{\tilde{j}_{|\mathcal{N}_{\tilde{i}}|}}(s))|\\&+|k_{\tilde{i},\mathrm{pr}_{\tilde{i}}(\bar{l})}(s,x_{\tilde{i}}(s),x_{\tilde{j}_1}(s),\dots,x_{\tilde{j}_{|\mathcal{N}_{\tilde{i}}|}}(s);x_{\tilde{i}0})|]ds\\\le&\int_0^\tau(M+v_{\max})ds<\delta t(M+v_{\max})=R_{\max}\end{aligned} \quad (15)$$

In order to finish the proof we exploit the following fact whose proof is rather straightforward. **Fact:** For every $x\in\partial(S+B(R))$, where $\emptyset\ne S\subset\mathbb{R}^n$ and $R>0$, it holds $|x-y|\ge R,\forall y\in S$. By exploiting the above fact with $S=S_{\bar{l}_{\tilde{i}}}$, $R=R_{\max}$, $y=x_{\tilde{i}0}$ and $x=x_{\tilde{i}}(\tau)$ we deduce from (15) that $x_{\tilde{i}}(\tau)\notin\partial(S_{\bar{l}_{\tilde{i}}}+B(R_{\max}))$ which contradicts the inclusion in (14) and the proof is complete. ■

IV. Admissible Space-Time Discretizations

We proceed by providing some extra assumptions for the dynamics as determined by the feedback law in (2). In particular we assume that the f_i's are globally Lipschitz functions. Furthermore, if we want to achieve more accurate bounds for the dynamics of the feedback controllers we assign to the free inputs v_i (those will be clarified in the proof of Proposition 4.1), we can choose (possibly) different Lipschitz constants $L_1,L_2>0$ such that for all $x_i,y_i\in\mathbb{R}^n$, $(x_{j_1},\dots,x_{j_{|\mathcal{N}_i|}}),(y_{j_1},\dots,y_{j_{|\mathcal{N}_i|}})\in\mathbb{R}^{|\mathcal{N}_i|n}$ and $i=1,\dots,N$ it holds

$$\begin{aligned}&|f_i(x_i,x_{j_1},\dots,x_{j_{|\mathcal{N}_i|}})-f_i(x_i,y_{j_1},\dots,y_{j_{|\mathcal{N}_i|}})|\\\le&L_1|(x_i,x_{j_1},\dots,x_{j_{|\mathcal{N}_i|}})-(x_i,y_{j_1},\dots,y_{j_{|\mathcal{N}_i|}})|, \quad (16)\\&|f_i(x_i,x_{j_1},\dots,x_{j_{|\mathcal{N}_i|}})-f_i(y_i,x_{j_1},\dots,x_{j_{|\mathcal{N}_i|}})|\\\le&L_2|(x_i,x_{j_1},\dots,x_{j_{|\mathcal{N}_i|}})-(y_i,x_{j_1},\dots,x_{j_{|\mathcal{N}_i|}})| \quad (17)\end{aligned}$$

In order to provide some extra informal motivation on considering both constants L_1 and L_2, we note that in order to derive sufficient conditions for a well posed discretization, we design for each agent i inside a cell S_{l_i} a feedback, in order to "track" a given reference trajectory (of i) starting in the same cell. In particular, the constant L_1 provides bounds on our choice of feedback in order to compensate for the deviation of agent's i dynamics from its corresponding dynamics along the reference trajectory, due to the time evolution of its neighbors' states. On the other hand, the constant L_2 provides bounds on our choice of feedback in order to compensate for the deviation of the initial state with respect to the initial state of the reference trajectory.

In order to apply the previous results it is useful to consider the least upper bound on the diameter of the cells in $\mathcal{S}$, namely $d_{\max} := \sup\{\sup\{|x-y| : x, y \in S_l\} : l \in \mathcal{I}\}$, which due to Definition 3.1 is well defined. We will call $d_{\max}$ the **diameter** of the cell decomposition.

Consider again system (1)-(2). We want to determine sufficient conditions relating the Lipschitz constants L_1, L_2, and the bounds M, $v_{\max}$ for the system's dynamics, as well as the space and time scales $d_{\max}$ and δt of the space-time discretization $\mathcal{S}-\delta t$ which guarantee that $\mathcal{S}-\delta t$ is well posed. As discussed at the beginning of the previous section, we require that the bound on the $f_i(\cdot)$ terms is greater than the maximum magnitude of the free inputs and thus impose the additional restriction

$$v_{\max} < M \tag{18}$$

According to Definition 3.3 establishment of a well posed discretization is based on the design of appropriate feedback laws which guarantee well posed transitions for all agents and their possible cell configurations. We proceed by defining the control laws we exploit in order to derive well posed discretizations. Consider a cell decomposition $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ of $\mathbb{R}^n$ and a time step δt. For each agent $i \in \{1,\ldots,N\}$ and cell configuration $\tilde{l}_i = (l_i, l_i^1, \ldots, l_i^{|\mathcal{N}_i|}) \in \mathcal{I}^{|\mathcal{N}_i|+1}$ of i let

$$(x_{i,G}, x_{j_1,G}, \ldots, x_{j_{|\mathcal{N}_i|},G}) \in S_{l_i} \times S_{l_i^1} \times \cdots \times S_{l_i^{|\mathcal{N}_i|}} \tag{19}$$

be an arbitrary reference point and define the feedback law $v_i = k_{i,\tilde{l}_i} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{(|\mathcal{N}_i|+1)n} \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$\begin{aligned} k_{i,\tilde{l}_i}(t, x_i, x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}}; x_{i0}) := k_{i,\tilde{l}_i,1}(x_i, x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}}) \\ + k_{i,\tilde{l}_i,2}(x_{i0}) + k_{i,\tilde{l}_i,3}(t; x_{i0}) \end{aligned} \tag{20}$$

where

$$\begin{aligned} k_{i,\tilde{l}_i,1}(x_i, x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}}) &:= -[f_i(x_i, x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}}) \\ &\quad - f_i(x_i, x_{j_1,G}, \ldots, x_{j_{|\mathcal{N}_i|},G})], \end{aligned} \tag{21}$$

$$k_{i,\tilde{l}_i,2}(x_{i0}) := -\tfrac{1}{\delta t}[x_{i0} - x_{i,G}], \tag{22}$$

$$\begin{aligned} k_{i,\tilde{l}_i,3}(t; x_{i0}) := -\Big[\tilde{f}_{i,\tilde{l}_i}\left(\tilde{x}_i(t) + \left(1 - \tfrac{t}{\delta t}\right)(x_{i0} - x_{i,G})\right) \\ - \tilde{f}_{i,\tilde{l}_i}(\tilde{x}_i(t))\Big] \end{aligned} \tag{23}$$

the function $\tilde{f}_{i,\tilde{l}_i}(\cdot)$ is given as

$$\tilde{f}_{i,\tilde{l}_i}(x_i) := f_i(x_i, x_{j_1,G}, \ldots, x_{j_{|\mathcal{N}_i|},G}), \forall x_i \in \mathbb{R}^n \tag{24}$$

and $\tilde{x}_i(\cdot)$ is the solution of the initial value problem

$$\dot{\tilde{x}}_i = \tilde{f}_{i,\tilde{l}_i}(\tilde{x}_i), \tilde{x}_i(0) = x_{i,G} \tag{25}$$

As we shall prove in the sequel, the solution $\tilde{x}_i(\cdot)$ of (25) is well defined and hence also the mapping $k_{i,\tilde{l}_i}(\cdot)$. We are now in position to provide the desired sufficient conditions for a well posed discretization.

Proposition 4.1: Consider a cell decomposition $\mathcal{S}$ of $\mathbb{R}^n$ with diameter $d_{\max}$, a time step δt, and assume that $d_{\max}$ and δt satisfy the restrictions

$$d_{\max} \in \left(0, \tfrac{v_{\max}^2}{4M\tilde{L}}\right] \tag{26}$$

$$\delta t \in \left[\tfrac{v_{\max} - \sqrt{v_{\max}^2 - 4M\tilde{L}d_{\max}}}{2M\tilde{L}}, \tfrac{v_{\max} + \sqrt{v_{\max}^2 - 4M\tilde{L}d_{\max}}}{2M\tilde{L}}\right] \tag{27}$$

with

$$\tilde{L} := \max\{2L_2 + 4L_1\sqrt{|\mathcal{N}_i|}, i = 1, \ldots, N\} \tag{28}$$

and where L_1 and L_2 are given in (16) and (17). Then the space-time discretization $\mathcal{S}-\delta t$ is well posed for the multi-agent system (1)-(2).

In particular, for each agent $i \in \{1,\ldots,N\}$ and cell configuration $\tilde{l}_i = (l_i, l_i^1, \ldots, l_i^{|\mathcal{N}_i|}) \in \mathcal{I}^{|\mathcal{N}_i|+1}$ of i we select any reference point $(x_{i,G}, x_{j_1,G}, \ldots, x_{j_{|\mathcal{N}_i|},G})$ as in (19) and consider the control law $k_{i,\tilde{l}_i}(\cdot)$ as determined by (20)-(25). Then the feedback law $k_{i,\tilde{l}_i}(\cdot)$ satisfies Property (P) and guarantees existence of a cell index $l_i' \in \mathcal{I}$, such that $l_i \xrightarrow{\tilde{l}_i} l_i'$ is well posed.

Proof: In order to prove the result, we want to show that the requirements of Definition 3.3 are fulfilled. Let $\mathcal{S} = \{S_l\}_{l\in\mathcal{I}}$ be a cell decomposition of $\mathbb{R}^n$ with diameter $d_{\max}$ and consider a time step δt, such that (26) and (27) hold. We want to show that for each $i = 1,\ldots,N$ and $\tilde{l}_i = (l_i, l_i^1, \ldots, l_i^{|\mathcal{N}_i|}) \in \mathcal{I}^{|\mathcal{N}_i|+1}$ there exists a cell index $l_i' \in \mathcal{I}$ such that the transition $l_i \xrightarrow{\tilde{l}_i} l_i'$ is well posed with respect to $\mathcal{S}-\delta t$. Pick $i \in \{1,\ldots,N\}$ and $\tilde{l}_i = (l_i, l_i^1, \ldots, l_i^{|\mathcal{N}_i|}) \in \mathcal{I}^{|\mathcal{N}_i|+1}$. In order to find $l_i' \in \mathcal{I}$ such that $l_i \xrightarrow{\tilde{l}_i} l_i'$ is well posed, we need according to Definition 3.3(a) to find a feedback law (6) satisfying Property (P) and in such a way that condition (C) is fulfilled. We break the proof in three steps.

STEP 1: Selection of the feedback $k_{i,\tilde{l}_i}(\cdot)$ and estimation of bounds on $k_{i,\tilde{l}_i,1}(\cdot)$, $k_{i,\tilde{l}_i,2}(\cdot)$ and $k_{i,\tilde{l}_i,3}(\cdot)$ as given in (21)-(23). In this step, we select an arbitrary reference point $(x_{i,G}, x_{j_1,G}, \ldots, x_{j_{|\mathcal{N}_i|},G})$ as in (19) and define $k_{i,\tilde{l}_i,1}(\cdot)$, $k_{i,\tilde{l}_i,2}(\cdot)$ and $k_{i,\tilde{l}_i,3}(\cdot)$ as in (21), (22) and (23), respectively. We next show that

$$\begin{aligned} &|k_{i,\tilde{l}_i,1}(x_i, x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}})| \leq L_1\sqrt{|\mathcal{N}_i|}(R_{\max} + d_{\max}), \\ &\forall x_i \in \mathbb{R}^n, x_{j_\kappa} \in S_{l_\kappa} + B(R_{\max}), \kappa = 1, \ldots, |\mathcal{N}_i| \end{aligned} \tag{29}$$

Indeed, let $(x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}}) \in \mathbb{R}^{|\mathcal{N}_i|n}$ satisfying $x_{j_\kappa} \in S_{l_\kappa} + B(R_{\max}), \kappa = 1, \ldots, |\mathcal{N}_i|$. Then for each $\kappa = 1, \ldots, |\mathcal{N}_i|$ there exists $\tilde{x}_{j_\kappa}$ with $\tilde{x}_{j_\kappa} \in S_{l_\kappa}$ and $|\tilde{x}_{j_\kappa} - x_{j_\kappa}| \leq R_{\max}$. The latter in conjunction with (16) and (21) imply that $|k_{i,\tilde{l}_i,1}(x_i, x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}})| \leq L_1|(x_{j_1} - x_{j_1,G}, \ldots, x_{j_{|\mathcal{N}_i|}} - x_{j_{|\mathcal{N}_i|},G})| \leq L_1\left(\sum_{\kappa=1}^{|\mathcal{N}_i|}(|x_{j_\kappa} - \tilde{x}_{j_\kappa}| + |\tilde{x}_{j_\kappa} - x_{j_\kappa,G}|)^2\right)^{\frac{1}{2}} \leq L_1\left(\sum_{\kappa=1}^{|\mathcal{N}_i|}(R_{\max} + d_{\max})^2\right)^{\frac{1}{2}} = L_1\sqrt{|\mathcal{N}_i|}(R_{\max} + d_{\max})$ and hence, that (29) holds. Furthermore, by recalling that $x_{i,G} \in S_{l_i}$, it follows directly from (22) that

$$|k_{i,\tilde{l}_i,2}(x_{i0})| \leq \tfrac{1}{\delta t}d_{\max}, \forall x_{i0} \in S_{l_i} \tag{30}$$

In the sequel, consider $\tilde{f}_{i,\tilde{l}_i}(\cdot)$ as given in (24) and notice, that due to (17), it satisfies the Lipschitz condition

$$|\tilde{f}_{i,\tilde{l}_i}(x_i) - \tilde{f}_{i,\tilde{l}_i}(y_i)| \leq L_2|x_i - y_i|, \forall x_i, y_i \in \mathbb{R}^n \tag{31}$$

By virtue of (31), the initial value problem (25) has a unique solution $\tilde{x}_i(\cdot)$ which exists for all $t \geq 0$ and thus $k_{i,\tilde{l}_i,3}(\cdot)$ is well defined. Hence, we obtain from (23) and (31) that

$$|k_{i,\tilde{l}_i,3}(t;x_{i0})| \leq L_2 d_{\max}, \forall t \in [0,\delta t], x_{i0} \in S_{l_i} \quad (32)$$

STEP 2: Verification of Property (P) for the feedback law (20) **for** $d_{\max} - \delta t$ **satisfying** (26) **and** (27)**.** In this step we prove that the proposed feedback law (20) satisfies Properties (P1) and (P2). Notice that (P1) follows from (20)-(23) and the Lipschitz property for the functions $f_i(\cdot)$. We next show that (P2) also holds. By taking into account (20), (29), (30) and (32) it suffices to prove that $L_1\sqrt{|\mathcal{N}_i|}(R_{\max} + d_{\max}) + \frac{1}{\delta t} d_{\max} + L_2 d_{\max} \leq v_{\max}$. By recalling (5), imposing the additional requirement that

$$\delta t(M + v_{\max}) \geq d_{\max} \Rightarrow R_{\max} \geq d_{\max} \quad (33)$$

and taking into account (18), it suffices instead to show that $M(2L_2 + 4L_1\sqrt{|\mathcal{N}_i|})\delta t^2 - v_{\max}\delta t + d_{\max} \leq 0$, which by virtue of (28), follows from

$$M\tilde{L}\delta t^2 - v_{\max}\delta t + d_{\max} \leq 0 \quad (34)$$

Thus, from (26), (27) and elementary calculations, we deuce that both (33) and (34) hold, and thus, that (P2) is satisfied.
STEP 3: Selection of the cell index l_i' and verification of Condition (C). Let $\tilde{x}_i(\cdot)$ be the solution of the reference trajectory as given by (25) and $l_i' \in \mathcal{I}$ the index of a cell $S_{l_i'}$ with $\tilde{x}_i(\delta t) \in S_{l_i'}$. We prove that for any initial cell configuration $\bar{l} = (\bar{l}_1, \ldots, \bar{l}_N) \in \mathcal{I}^N$ with $\mathrm{pr}_i(\bar{l}) = \tilde{l}_i$, selection of feedback laws in (7) which satisfy Property (P) for all $\hat{i} \in \{1, \ldots, N\} \setminus \{i\}$, and for each initial condition $x_{i0} \in S_{l_i}$ of i and $x_{\hat{i}0} \in S_{\bar{l}_{\hat{i}}}$ of the other agents $\hat{i} \in \{1, \ldots, N\} \setminus \{i\}$, the solution of the closed loop system (1)-(2)-(20)-(7) is defined for all $t \in [0, \delta t]$ and the trajectory $x_i(\cdot)$ of agent i at δt satisfies $x_i(\delta t) = \tilde{x}_i(\delta t)$, namely, coincides with the endpoint of the reference trajectory. We first note that due to the result of Proposition 3.5, the solution of the closed loop system is defined on the whole interval $[0, \delta t]$. In order to show that $x_i(\delta t) = \tilde{x}_i(\delta t)$, we show that $x_i(\cdot)$ is an appropriate modification of the reference trajectory $\tilde{x}_i(\cdot)$. In particular, we will show that

$$x_i(t) = \tilde{x}_i(t) + \left(1 - \tfrac{t}{\delta t}\right)(x_{i0} - x_{i,G}), \forall t \in [0, \delta t] \quad (35)$$

holds, which then implies the desired result. Indeed, from (1)-(2), (20), (21), (24) and (25) we have that $\dot{\tilde{x}}_i(t) = \tilde{f}_{i,\tilde{l}_i}(\tilde{x}_i(t))$; $\dot{x}_i(t) = f_i(x(t)) + k_{i,\tilde{l}_i}(t, x_i(t), \bar{x}(t); x_{i0}) = \tilde{f}_{i,\tilde{l}_i}(x_i(t)) + k_{i,\tilde{l}_i,2}(x_{i0}) + k_{i,\tilde{l}_i,3}(t; x_{i0})$ and hence, that $\tilde{x}_i(t) = x_{i,G} + \int_0^t \tilde{f}_{i,\tilde{l}_i}(\tilde{x}_i(s))ds$; $x_i(t) = x_{i0} + \int_0^t (\tilde{f}_{i,\tilde{l}_i}(x_i(s)) + k_{i,\tilde{l}_i,2}(x_{i0}) + k_{i,\tilde{l}_i,3}(s; x_{i0}))ds$. Then, it follows from (22) and (23) that $x_i(t) - \tilde{x}_i(t) = x_{i0} - x_{i,G} + \int_0^t [\tilde{f}_{i,\tilde{l}_i}(x_i(s)) - \tilde{f}_{i,\tilde{l}_i}(\tilde{x}_i(s)) + k_{i,\tilde{l}_i,2}(x_{i0}) + k_{i,\tilde{l}_i,3}(s; x_{i0})]ds = \left(1 - \frac{t}{\delta t}\right)(x_{i0} - x_{i,G}) + \int_0^t [\tilde{f}_{i,\tilde{l}_i}(x_i(s)) - \tilde{f}_{i,\tilde{l}_i}(\tilde{x}_i(s) + (1 - \frac{s}{\delta t})(x_{i0} - x_{i,G}))]ds, \forall t \in [0, \delta t]$. Hence, we get from (31) that $|x_i(t) - \tilde{x}_i(t) - \left(1 - \frac{t}{\delta t}\right)(x_{i0} - x_{i,G})| \leq \int_0^t L_2 \left| x_i(s) - \tilde{x}_i(s) - \left(1 - \frac{s}{\delta t}\right)(x_{i0} - x_{i,G})\right| ds, \forall t \in [0, \delta t]$. Application of the Gronwall Lemma implies that (35) holds and thus, that $x_i(\delta t) = \tilde{x}_i(\delta t)$ as desired. ∎

V. Conclusions

We have provided a decentralized abstraction framework in order to extract discrete state transition systems for multi-agent systems under coupled constraints and quantified admissible space-time discretizations which allow for well posed abstractions. The abstraction setup is focused on the single integrator case and relies on the design of hybrid control laws which take into account the agents' feedback interconnection and ensure feasibility of discrete transitions.

The approach of Proposition 4.1 can be modified in order to provide sufficient conditions which guarantee that each agent can reach multiple discrete cells in time δt. Thus, the corresponding hybrid controllers and the result of Proposition 3.4 can be exploited for motion planning, by specifying multiple transition possibilities for each agent through the selection of the feedback laws that are assigned to the free inputs. We also intend to extend the results for the case of bounded domains in order to obtain finite transition systems.

Further research directions include the study of system theoretic properties for the proposed hybrid control scheme (see [7, Section 1.2.5]) and the generalization of the decentralized abstraction methodology through an event-based online discretization framework. The latter should result in an updated choice of $d_{\max}$ and δt and significantly reduce computational requirements for high level task specifications.

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