A connection between formation infeasibility and velocity alignment in kinematic multi-agent systems ⋆

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Abstract

In this paper, a feedback control strategy that achieves convergence of a multi-agent system to a desired formation configuration is proposed for both the cases of agents with single integrator and nonholonomic unicycle-type kinematics. When inter-agent objectives that specify the desired formation cannot occur simultaneously in the state space the desired formation is infeasible. It is shown that under certain assumptions, formation infeasibility forces the agents' velocity vectors to a common value at steady state. This provides a connection between formation infeasibility and flocking behavior for the multi-agent system. We finally also obtain an analytic expression of the common velocity vector in the case of formation infeasibility.

Key words: Decentralized Control; Autonomous Systems; Formation Control; Multi-agent Systems.

1 Introduction

The emerging use of large-scale multi-robot and multi-vehicle systems in various modern applications has raised the need for the design of control laws that force a team of multiple vehicles/robots (from now on called "agents") to achieve various goals. As the number of agents increases, decentralized control approaches are preferable to centralized ones, due to the fact that they respect the limited communication and sensing capabilities of the agents and moreover provide a reduction in the computational complexity of the applied algorithms. There are various objectives that the control design aims at achieving in the case of a multi-agent team. In the case of formation control, agents must converge to a desired configuration encoded by the inter-agent relative positions. Many feedback control schemes that achieve formation stabilization to a desired formation in a distributed manner have been proposed in literature (see for example Dimarogonas et al. (2006),Lin et al. (2005) for some recent efforts). The so-called agreement problem, where agents must converge to the same point in the state space (Olfati-Saber and Murray (2004),Ji and Egerstedt (2005),Ren et al. (2004)) is also relevant. On the other hand, flocking behavior involves, among others, convergence of the velocity vectors and orientations of the agents to a common value at steady state; relevant contributions include Jadbabaie et al. (2003), Tanner et al. (2003),Olfati-Saber (2006).

In this paper a formation control strategy is adopted for both the cases of single integrator and nonholonomic unicycle-type kinematic agents. In the nonholonomic case, we propose a discontinuous and time invariant control law, that drives the unicycle team to desired final relative positions, imposed by the formation specification at hand. When these specifications cannot occur simultaneously in the state space, the desired formation is rendered infeasible. In this case, it is shown that under certain assumptions the formation infeasibility forces the agents velocity vectors to a common value at steady state, in both the cases of nonholonomic and single integrator kinematic agents. An analytic expression of the common velocity vector is then provided as a function of the desired inter-agent position vectors.

We should stress that although the single integrator case of the formation control problem could be consid-

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ered as related to the development of Fax and Murray (2002), the formulation of this paper is such that it allows to prove the result connecting formation infeasibility with velocity alignment. Geometric conditions for the case of formation infeasibility have been examined in the context of graph rigidity (Hendrickx et al. (2005)), graph controllability (Mesbahi (2005)), sensor networks (de Silva and Ghrist (2007)), and connectivity graphs (Muhammad and Egerstedt (2005)). Moreover, a formation control strategy for nonholonomic agents also recently appeared in Lin et al. (2005). The authors of that work use a time varying nonholonomic control strategy, which includes a periodic open loop averaging control law for the angular velocity. Generally, this type of control law provides worse convergence results compared to the proposed discontinuous and time invariant control law. Our preference to time-invariant strategies is further established in Kim and Tsiotras (2002), where the authors provide both experimental and theoretical comparisons between these two types of nonholonomic controllers. One of the main results of that development was that time varying controllers are too slow and oscillatory for most practical situations, while time-invariant controllers exhibit a significantly better behavior.

The rest of this paper is organized as follows: Section 2 reviews some mathematical tools used in the sequel. In the next two sections the results regarding formation control and the connection between formation infeasibility and velocity alignment are presented. Section 3 involves the case of single integrator, while Section 4 the case of nonholonomic unicycle-type agents. In Section 5 simulation results are presented to support the previous results. A summary of the results of this paper is given in Section 6.

2 Mathematical Preliminaries

2.1 Tools from Algebraic Graph Theory

In this subsection we review some tools from algebraic graph theory that we shall use in the stability analysis of the next sections. The following can be found in any standard textbook on algebraic graph theory(e.g., Godsil and Royle (2001)).

For an undirected graph G with n vertices the *adjacency matrix* $A = A(G) = (a_{ij})$ is the $n \times n$ matrix given by $a_{ij} = 1$, if $(i, j) \in E$, where E the set of edges of G, and $a_{ij} = 0$, otherwise. If there is an edge connecting two vertices i, j, i.e. $(i, j) \in E$, then i, j are called *adjacent*. A *path* of length r from a vertex i to a vertex j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of the graph G, then G is called *connected* (otherwise it is called *disconnected*). The *degree* d_i of vertex i is defined as the number of its neighboring vertices, i.e. $d_i = \{\#j : (i, j) \in E\}$. Let Δ be the $n \times n$ diagonal matrix of d_i's. The (combinatorial) *Laplacian* of G is the symmetric positive semidefinite matrix $\mathcal{L} = \Delta - A$. The Laplacian captures many interesting topological properties of the graph. Of particular interest in our case is the fact that for a connected graph, the Laplacian has a single zero eigenvalue and the corresponding eigenvector is the vector of ones, $\overrightarrow{\mathbf{1}}$.

2.2 Tools from Nonsmooth Analysis

We now review some elements from nonsmooth analysis and Lyapunov theory for nonsmooth systems that we use in the stability analysis of the next sections.

For a differential equation with discontinuous right-hand side we have the following definition:

Definition 1 *(Filippov (1988)) In the case when the state-space is finite dimensional, the vector function $x(.)$ is called a* Filippov solution *of $\dot{x} = f(x)$ if it is absolutely continuous and $\dot{x} \in K[f](x)$ almost everywhere where $K[f](x) \equiv \overline{co}\{\lim_{x_i \to x} f(x_i) | x_i \notin N\}$, where N is a set of measure zero.*

The following chain rule provides a calculus for the time derivative of the energy function in the nonsmooth case:

Theorem 1 *(Shevitz and Paden (1994)) Let x be a Filippov solution to $\dot{x} = f(x)$ on an interval containing t and $V : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz and regular function. Then $V(x(t))$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and $\frac{d}{dt}V(x(t)) \in^{a.e.} \dot{\tilde{V}}(x) := \bigcap_{\xi \in \partial V(x(t))} \xi^T K[f](x(t))$, where "a.e." stands for "almost everywhere".*

In this theorem, ∂V is *Clarke's generalized gradient*. The definition of the generalized gradient and of the *regularity* of a function can be found in Clarke (1983). In the case encountered in this paper, the candidate Lyapunov function V we use is smooth and hence regular, while its generalized gradient is a singleton which is equal to its usual gradient everywhere in the state space: $\partial V(x) = \{\nabla V(x)\} \forall x$.

We shall use the following nonsmooth version of LaSalle's invariance principle to prove the convergence of the prescribed system:

Theorem 2 *(Shevitz and Paden (1994)) Let Ω be a compact set such that every Filippov solution to the autonomous system $\dot{x} = f(x), x(0) = x(t_0)$ starting in Ω is unique and remains in Ω for all $t \geq t_0$. Let $V : \Omega \to \mathbb{R}$ be a time independent regular function such that $v \leq 0 \forall v \in \dot{\tilde{V}}$ (if $\dot{\tilde{V}}$ is the empty set then this is trivially satisfied). Define $S = \{x \in \Omega | 0 \in \dot{\tilde{V}}\}$. Then every trajectory in Ω converges to the largest invariant set, M, in the closure of S.*

3 Single Integrator Agents

This part of the paper is devoted to the relatively simple case of formation design for multiple single integrator agents. We first provide a control law that drives the agents to a feasible formation configuration. We then show that formation infeasibility results in velocity alignment for the multi-agent team.

3.1 System and Problem Definition

Consider a system of N point agents operating in $\mathbb{R}^2$. Let $q_i \in \mathbb{R}^2$ denote the position of agent i. We denote by $q = [q_1, \ldots, q_N]^T$ the stack vector of all agents positions. The motion of each agent is described by the single integrator:

$$\dot{q}_i = u_i, i \in \mathcal{N} = [1, \ldots, N] \tag{1}$$

where u_i denotes the velocity (control input) of agent i.

Each agent's objective is to converge to a desired relative configuration with respect to a certain subset of the rest of the team, in a manner that will lead the whole team to a desired formation. Specifically, each agent is assigned with a specific subset N_i of the rest of the team, called agent i's *communication set* with which it can communicate in order to achieve the desired formation. The desired formation can be encoded in terms of an undirected graph, from now on called the *formation graph* $G = \{V, E\}$, whose set of vertices $V = \{1, ..., N\}$ is indexed by the team members, and whose set of edges $E = \{(i,j) \in V \times V | j \in N_i\}$ contains pairs of vertices that represent inter-agent formation specifications. A vector $c_{ij} \in \mathbb{R}^2$ is associated to each edge $(i,j) \in E$, in order to specify the desired inter-agent relative positions in the final formation configuration.

The objective of each agent i is to be stabilized in a desired relative position c_{ij} with respect to each member j of N_i. Each agent has only knowledge of the relative displacement of agents that belong to its communication set. This fact highlights the decentralized nature of the approach. We assume moreover that the formation graph is undirected, in the sense that $i \in N_j \Leftrightarrow j \in N_i, \forall i,j \in \mathcal{N}, i \neq j$. It is obvious that $(i,j) \in E$ iff $i \in N_j \Leftrightarrow j \in N_i$. Formation feasibility is defined as follows:

Definition 2 *The formation configuration is called* feasible *if the set*

$$\Phi \triangleq \left\{ q \in \mathbb{R}^{2N} \left| q_i - q_j = c_{ij}, \, \forall (i,j) \in E \right. \right\}$$

of feasible formation configurations is nonempty.

Whenever the latter does not hold, the formation configuration is called *infeasible*.

3.2 Control Strategy for Feasible Formation

We propose the following control law for agent i:

$$u_i = -\frac{\partial \gamma_i}{\partial q_i} \tag{2}$$

where

$$\gamma_i = \frac{1}{2} \sum_{j \in N_i} \|q_i - q_j - c_{ij}\|^2 \tag{3}$$

The following theorem examines the convergence of the system to the desired formation configuration:

Theorem 3 *Assume that the formation configuration is feasible and that the formation graph is connected. Then, under the control law (2) the state of the system converges to the desired formation configuration.*

Proof: Differentiating γ_i with respect to q_i we have

$$\frac{\partial \gamma_i}{\partial q_i} = \sum_{j \in N_i} (q_i - q_j - c_{ij}) = \sum_{j \in N_i} (q_i - q_j) + c_{ii}$$

where $c_{ii} = -\sum_{j \in N_i} c_{ij}$. Using now (1),(2), we can then easily compute

$$\dot{q} = \left[-\frac{\partial \gamma_1}{\partial q_1} \; \ldots \; -\frac{\partial \gamma_N}{\partial q_N} \right]^T = -(Lq + c_l) \tag{4}$$

where $c_l = [c_{11}, \ldots, c_{NN}]^T$, and where $L = \mathcal{L} \otimes I_2$ and $\otimes$ denotes Kronecker product, as usual. The $N \times N$ matrix $\mathcal{L}$ is the Laplacian of the formation graph. We use $V = \sum_i \gamma_i$ as a candidate Lyapunov function. We can then easily derive that

$$\sum_i \nabla \gamma_i = 2(Lq + c_l) \tag{5}$$

The time-derivative of V is now computed:

$$\begin{aligned} V = \sum_i \gamma_i \Rightarrow \dot{V} = \left\{ \sum_i (\nabla \gamma_i)^T \right\} \cdot \dot{q} = \\ = -2(Lq + c_l)^T (Lq + c_l) = -2 \|Lq + c_l\|^2 \end{aligned}$$

The level sets of V define compact sets with respect to the agents' relative positions. In this way, we can apply LaSalle's invariance principle for the closed loop system. Specifically, for all $(i,j) \in E$ we have

$$\begin{aligned} & V \le c \Rightarrow \gamma_i \le c \Rightarrow \tfrac{1}{2} \|q_i - q_j - c_{ij}\|^2 \le c \Rightarrow \\ & \|q_i - q_j - c_{ij}\| \le \sqrt{2c} \Rightarrow |\|q_i - q_j\| - \|c_{ij}\|| \le \sqrt{2c} \Rightarrow \\ & \Rightarrow -\sqrt{2c} + \|c_{ij}\| \le \|q_i - q_j\| \le \sqrt{2c} + \|c_{ij}\| \Rightarrow \\ & \Rightarrow 0 \le \|q_i - q_j\| \le \sqrt{2c} + c_{\max} \end{aligned}$$

where $c_{\max} \triangleq \max_{(i,j)\in E} \|c_{ij}\|$. Connectivity of the formation graph ensures that the maximum length of a path connecting two vertices of the graph is at most $N-1$. Hence $0 \leq \|q_i - q_j\| \leq \left(\sqrt{2c} + c_{\max}\right)(N-1)\,,\ \forall i,j \in \mathcal{N}$. Application of LaSalle's invariance principle ensures the convergence of the system to the largest invariant subset of the set $S = \{q : Lq + c_l = 0\}$.

For all $i \in \mathcal{N}$, let c_i denote the configuration of agent i in a desired formation configuration with respect to the global coordinate frame. It is then obvious that $c_{ij} = c_i - c_j\ \forall (i,j) \in E$ for all possible desired final formations. Define $q_i - q_j - c_{ij} = q_i - q_j - (c_i - c_j) = \tilde{q}_i - \tilde{q}_j$. Then the feasibility assumption implies that $Lq + c_l = 0 \Rightarrow L\tilde{q} = 0 \Rightarrow \mathcal{L}\tilde{x} = \mathcal{L}\tilde{y} = 0$ where $\tilde{x}, \tilde{y}$ the stack vectors of $\tilde{q}$ in the x, y directions. The fact that the formation graph is connected implies that the Laplacian has a simple zero eigenvalue with corresponding eigenvector the vector of ones, $\overrightarrow{\mathbf{1}}$. This guarantees that both $\tilde{x}, \tilde{y}$ are eigenvectors of $\mathcal{L}$ belonging to span$\{\overrightarrow{\mathbf{1}}\}$. Therefore all $\tilde{q}_i$ are equal to a common vector value q^*. Hence $\tilde{q}_i = q^*\ \forall i \Rightarrow q_i - q_j = c_{ij}\ \forall i,j \in N_i$. We conclude that the agents converge to the desired relative configuration. $\diamondsuit$

3.3 Formation infeasibility results in velocity alignment

The key assumption behind the stability analysis of the previous section is *formation feasibility*, in the sense discussed in Section 2. But what happens when the formation configuration is infeasible, i.e. there does *not* exist such a configuration in the state space? The answer is given in the next theorem:

Theorem 4 *If the formation graph is connected, the system reaches a configuration in which all agents have the same velocity vectors.*

Proof: Differentiating equation(4) with respect to time we get

$$\dot{q} = -\left(Lq + c_l\right) \Rightarrow \ddot{q} = -L\dot{q} \tag{6}$$

Using $W = \frac{1}{2}\|\dot{q}\|^2$ as a candidate Lyapunov function for system (6) and taking its time derivative we have

$$W = \frac{1}{2}\|\dot{q}\|^2 \Rightarrow \dot{W} = \dot{q}^T\ddot{q} = -\dot{q}^T L\dot{q} \leq 0,$$

since L is positive semidefinite. LaSalle's Invariance Principle guarantees that the state of the system converges to the largest invariant subset of the set $S = \left\{\dot{q} | \dot{W} = 0\right\}$. Since $\ddot{q} = -L\dot{q}$ we necessarily have $\ddot{q} = 0$ inside S. Hence agent velocities converge to a constant value. Denoting by v_x, v_y the N-dimensional stack vectors of the components of the agents' velocities in the x, y directions at steady state, we have

$$\dot{W} = 0 \Rightarrow \dot{q}^T\left(\mathcal{L} \otimes I_2\right)\dot{q} = 0 \Rightarrow v_x^T\mathcal{L}v_x + v_y^T\mathcal{L}v_y = 0$$

at steady state. This implies that both v_x, v_y are eigenvectors of $\mathcal{L}$ corresponding to the zero eigenvalue, meaning that v_x, v_y belong to span$\{\overrightarrow{\mathbf{1}}\}$, which ensures that all agent velocity vectors will have the same components at steady state, and will therefore be equal. $\diamondsuit$

What is left is to provide an analytic expression of the common velocity vector that the agents reach. This is the result of the following corollary:

Corollary 5 *Assume that the undirected formation graph is connected. Then the agents attain a common velocity vector $\dot{q}_i = \dot{q}^*$ for all $i \in \mathcal{N}$ which is given by*

$$\dot{q}^* = -\frac{1}{N}\sum_i c_{ii}.$$

Proof: The fact that the agents reach a common constant velocity is derived from Theorem 4. Denoting this common velocity by $\dot{q}^*$ and using the notation $\tilde{c}_l = \left[\dot{q}^*, \ldots, \dot{q}^*\right]^T$, equation (4) yields $\dot{q} = -\left(Lq + c_l\right) = \tilde{c}_l$. Hence $Lq + c_l = -\tilde{c}_l \Rightarrow Lq = -c_l - \tilde{c}_l$. The fact that the formation graph is undirected implies that the sum of the elements of each vector that belongs to the range of the corresponding Laplacian matrix is zero. In particular, the Laplacian of an undirected graph has zero row and column sums and hence we have

$$\overrightarrow{\mathbf{1}}^T\mathcal{L} = 0 \Rightarrow \left(\overrightarrow{\mathbf{1}}^T \otimes I_2\right)L = 0 \Rightarrow \left(\overrightarrow{\mathbf{1}}^T \otimes I_2\right)Lq = 0$$
$$\Rightarrow \left(\overrightarrow{\mathbf{1}}^T \otimes I_2\right)\left(-c_l - \tilde{c}_l\right) = 0$$

The last equation is equivalent to the fact the sum of the elements of the vector $-c_l - \tilde{c}_l$ is zero in both directions x, y. Thus,

$$Lq = -c_l - \tilde{c}_l \Rightarrow \sum_i\left(c_{ii} + \dot{q}^*\right) = 0 \Rightarrow \dot{q}^* = -\frac{1}{N}\sum_i c_{ii},$$

and this concludes the proof. $\diamondsuit$

The previous corollary reveals the fact that the norm of the common velocity vector is given by $\|\dot{q}^*\| = \frac{1}{N}\left\|\sum_i c_{ii}\right\|$. Hence the orientation and volume of the velocity of the resulting flock are completely determined by the number N of the team members and the term $\sum_i c_{ii}$.

4 Nonholonomic Agents

This section includes the nonholonomic counterpart of the results of the previous section. The problem formulation is the same as in the single integrator case, but agent motion is now described by the nonholonomic unicycle model.

Each of the N mobile agents has a specific orientation θ_i with respect to the global coordinate frame. The orientation vector of the agents is represented by $\theta = [\theta_1 \dots \theta_N]^T$. The configuration of each agent is represented by $p_i = \begin{bmatrix} q_i & \theta_i \end{bmatrix}^T \in \mathbb{R}^2 \times (-\pi, \pi]$. Agent motion is described by the following nonholonomic kinematics:

$$\begin{aligned} \dot{x}_i &= u_i \cos\theta_i \\ \dot{y}_i &= u_i \sin\theta_i \\ \dot{\theta}_i &= \omega_i \end{aligned} \tag{7}$$

for $i \in \mathcal{N} = [1, \dots, N]$ and where u_i, ω_i denote the translational and rotational velocity of agent i, respectively. We shall use the function γ_i, defined in equation (3), for each agent i. Hence each agent aims to converge to a desired relative position with respect to some members of the team. As previously, it is easy to derive that in the case of a feasible formation, equation (5) is still valid: $\sum\limits_i \nabla\gamma_i = 2(Lq + c_l)$, where $L = \mathcal{L} \otimes I_2$. In the analysis that follows, we use the decoupling of the stack vector $q = [x, y]^T$ and the vector $c_l = [c_x, c_y]^T$ into the coefficients that correspond to the x, y directions of the agents respectively. We use the function $\mathrm{sgn}(x) = 1$, if $x \geq 0$ and $\mathrm{sgn}(x) = -1$, otherwise. The function $\arctan 2(x, y)$ that is also used is the same as the arc tangent of the two variables x and y with the distinction that the signs of both arguments are used to determine the quadrant of the result. We also use $\arctan 2(0, 0) = 0$. Furthermore, the notation $(a)_i$ for a vector a denotes its i-th element.

4.1 Stability of a feasible formation

Theorem 6 *Assume that the formation configuration is feasible and that the formation graph is connected. Then the feedback control strategy:*

$$u_i = -\mathrm{sgn}\left\{\gamma_{xi}\cos\theta_i + \gamma_{yi}\sin\theta_i\right\} \cdot \left(\gamma_{xi}^2 + \gamma_{yi}^2\right)^{1/2} \tag{8}$$

$$\omega_i = -\left(\theta_i - \theta_{nh_i}\right) \tag{9}$$

where $\gamma_{xi} = (\mathcal{L}x + c_x)_i, \gamma_{yi} = (\mathcal{L}y + c_y)_i$ *and the "nonholonomic angle"* $\theta_{nh_i} = \arctan 2(\gamma_{yi}, \gamma_{xi})$ *drives the agents to the desired formation configuration with zero orientation.*

Proof: By using the positive semidefinite function $V = \sum\limits_i \gamma_i$ as a candidate Lyapunov function and computing its generalized time derivative we get

$$V = \sum_i \gamma_i \Rightarrow \dot{\tilde{V}} = \left(\sum_i \nabla\gamma_i\right)^T \cdot K \begin{bmatrix} u_1\cos\theta_1 \\ u_1\sin\theta_1 \\ \vdots \\ u_N\cos\theta_N \\ u_N\sin\theta_N \end{bmatrix}$$

$$\subset 2(Lq + c_l)^T \begin{bmatrix} K[u_1]\cos\theta_1 \\ K[u_1]\sin\theta_1 \\ \vdots \\ K[u_N]\cos\theta_N \\ K[u_N]\sin\theta_N \end{bmatrix}$$

$$\subset 2(\mathcal{L}x + c_x)^T \begin{bmatrix} K[u_1]\cos\theta_1 \\ \vdots \\ K[u_N]\cos\theta_N \end{bmatrix} +$$

$$+2(\mathcal{L}y + c_y)^T \begin{bmatrix} K[u_1]\sin\theta_1 \\ \vdots \\ K[u_N]\sin\theta_N \end{bmatrix}$$

$$\subset \sum_i \left\{2K[u_i]\left((\mathcal{L}x + c_x)_i\cos\theta_i + (\mathcal{L}y + c_y)_i\sin\theta_i\right)\right\}$$

where we used Theorem 1.3 in Paden and Sastry (1987) to calculate the inclusions of the Filippov set in the previous analysis. Since $K[\mathrm{sgn}(x)]x = \{|x|\}$(Paden and Sastry (1987),Theorem 1.7), the choice of control laws (8),(9) results in

$$\dot{\tilde{V}} = 2\sum_i \left\{-\left|\gamma_{xi}\cos\theta_i + \gamma_{yi}\sin\theta_i\right|\left(\gamma_{xi}^2 + \gamma_{yi}^2\right)^{1/2}\right\} \leq 0,$$

i.e., the generalized derivative reduces to a singleton. Using the same arguments as in the proof of the single integrator case, we deduce that the level sets of the candidate Lyapunov function are compact and invariant and therefore we can apply the (nonsmooth version of) LaSalle's invariance Principle.

The last inequality hence implies that the trajectories of the system converge to the largest invariant set contained in the set

$$S = \left\{(\gamma_{xi} = \gamma_{yi} = 0) \vee (\gamma_{xi}\cos\theta_i + \gamma_{yi}\sin\theta_i = 0), \forall i \in \mathcal{N}\right\}$$

However, for each $i \in \mathcal{N}$, we have $|\omega_i| = \frac{\pi}{2}$ whenever $\gamma_{xi}\cos\theta_i + \gamma_{yi}\sin\theta_i = 0$, due to the proposed angular velocity control law. In particular, this choice of angular velocity renders the surface $\gamma_{xi}\cos\theta_i + \gamma_{yi}\sin\theta_i = 0$ non-invariant for agent i (Tanner and Kyriakopoulos (2003)), whenever i is not located at the desired equilibrium, namely when $\gamma_{xi} = \gamma_{yi} = 0$. Hence the largest invariant set contained in S is $S \supset S_0 = \{\gamma_{xi} = \gamma_{yi} = 0, \forall i \in \mathcal{N}\}$. In this set, the orientations of all agents converge to zero, since $\theta_{nh_i} = 0$ for each agent i in S_0. For $(\gamma_{xi} = \gamma_{yi} = 0)$ we have $\theta_{nhi} = 0$ so that $\theta_i = 0 \ \forall \ i \in \mathcal{N}$. In addition $(\gamma_{xi} = \gamma_{yi} = 0)$ $\forall i$ guarantees that the agents converge to the desired formation configuration. This is easily derived by the fact that $(\gamma_{xi} = \gamma_{yi} = 0)$ $\forall i \Rightarrow Lq + c_l = 0$ and using the same arguments as in the last part of the proof of Theorem 3. $\diamondsuit$

4.2 Formation infeasibility results in velocity alignment

The previous section has established a connection between formation infeasibility and flocking behavior for the case of multiple agents with single integrator kinematics. In this section, we show that a similar result holds for the unicycle case as well.

The control law is now given by

$$u_i = -\mathrm{sgn}\left\{\gamma_{xi}\cos\theta_i + \gamma_{yi}\sin\theta_i\right\} \cdot \left(\gamma_{xi}^2 + \gamma_{yi}^2\right)^{1/2} \quad (10)$$

$$\omega_i = -\left(\theta_i - \theta_{nh_i}\right) + \dot{\theta}_{nh_i} \quad (11)$$

where the terms $\gamma_{xi}, \gamma_{yi}, \theta_{nh_i}$ were defined previously.

We should point out that the time derivative of θ_{nh_i} is not defined at $\gamma_{xi} = \gamma_{yi} = 0$. In implementation, one can use the modification of the nonholonomic angle used in Egerstedt and Hu (2001):

$$\hat{\theta}_{nh_i} = \begin{cases} \theta_{nh_i}, \ \text{if} \theta_{nh_i} > \varepsilon \\ \frac{\theta_{nh_i}\left(-2\rho_i^3 + 3\varepsilon\rho_i^2\right) + \theta_r\left(-2(\varepsilon-\rho)^3 + 3\varepsilon(\varepsilon-\rho)^2\right)}{\varepsilon^3}, \text{if} \theta_{nh_i} \leq \varepsilon \end{cases} \quad (12)$$

where $\rho_i = \sqrt{\gamma_{xi}^2 + \gamma_{yi}^2}$ and ε is chosen arbitrarily small. The following theorem contains the main result of this section.

Theorem 7 *If the formation graph is connected, then the feedback strategy (10),(11) drives the nonholonomic multi-agent system to a configuration in which all agents have the same velocities and orientations.*

Proof: Equation (11) implies that θ_i is aligned with θ_{nh_i} as $t \to \infty$. The closed loop kinematics for the x, y-coefficients then become

$$\dot{x}_i = u_i \cos\theta_{nh_i} = -\mathrm{sgn}\left\{\gamma_{xi}\cos\theta_{nh_i} + \gamma_{yi}\sin\theta_{nh_i}\right\}\gamma_{xi}$$
$$\dot{y}_i = u_i \sin\theta_{nh_i} = -\mathrm{sgn}\left\{\gamma_{xi}\cos\theta_{nh_i} + \gamma_{yi}\sin\theta_{nh_i}\right\}\gamma_{yi}$$

But since by definition of θ_{nh_i} we have $\gamma_{xi}\cos\theta_{nh_i} + \gamma_{yi}\sin\theta_{nh_i} > 0$, then at steady state the previous equations reduce to:

$$\begin{aligned} \dot{x}_i &= -\gamma_{xi} \\ \dot{y}_i &= -\gamma_{yi} \end{aligned} \quad (13)$$

for $i \in \mathcal{N} = \{1, ..., N\}$. Using $W = \frac{1}{2}\sum_i \left(\dot{x}_i^2 + \dot{y}_i^2\right)$ as a candidate Lyapunov function for the system (13) and differentiating with respect to time we get:

$$\dot{W} = \sum_i \left(\dot{x}_i\ddot{x}_i + \dot{y}_i\ddot{y}_i\right) = -\sum_i \left(\dot{x}_i\dot{\gamma}_{xi} + \dot{y}_i\dot{\gamma}_{yi}\right) =$$
$$-\sum_i \left(\dot{x}_i\left(L\dot{x}\right)_i + \dot{y}_i\left(L\dot{y}\right)_i\right) \Rightarrow \dot{W} = -\dot{x}^T L\dot{x} - \dot{y}^T L\dot{y} \leq 0$$

LaSalle's Invariance Principle guarantees that the state of the system (13) converges to the largest invariant subset of the set $S = \left\{\dot{q} | \dot{W} = 0\right\}$. Using the same arguments as in the holonomic case, we deduce that at steady state both $\dot{x} = \left[\dot{x}_1, ..., \dot{x}_N\right]^T, \dot{y} = \left[\dot{y}_1, ..., \dot{y}_N\right]^T$ are eigenvectors of $\mathcal{L}$ corresponding to the zero eigenvalue, meaning that $\dot{x}, \dot{y}$ belong to span$\{\overrightarrow{\mathbf{1}}\}$, which ensures that all agent velocity vectors will have the same components at steady state, and will therefore be equal. It is obvious then that the nonholonomic angles θ_{nh_i} of all agents are equal (since all γ_{xi}, γ_{yi} are equal) and the fact that $\theta_i = \theta_{nh_i} \forall i$ guarantees that at steady state all agents will have a common orientation. $\diamondsuit$

Similar arguments to the single integrator case provide the analytic expression of the common velocity vector in the nonholonomic case as well. In particular, since

$$\left[\begin{array}{cc} \frac{\partial\gamma_i}{\partial x_i} & \frac{\partial\gamma_i}{\partial y_i} \end{array}\right]^T = \frac{\partial\gamma_i}{\partial q_i} = \sum_{j \in N_i} \left(q_i - q_j\right) + c_{ii}$$

equation (13), written in stack vector form is equivalent to

$$\dot{q} = \left[\begin{array}{ccc} -\frac{\partial\gamma_1}{\partial q_1} & \dots & -\frac{\partial\gamma_N}{\partial q_N} \end{array}\right]^T = -\left(Lq + c_l\right) \quad (14)$$

Hence the nonholonomic system behaves as in the single integrator case in the velocity space. The previous discussion is summarized in the following corollary:

Corollary 8 *Let the multi-agent nonholonomic system be driven by the control law (10),(11). Assume that the undirected formation graph is connected. Then the agents attain a common velocity vector $\dot{q}_i = \dot{q}^*$ for all $i \in \mathcal{N}$ which is given by $\dot{q}^* = -\frac{1}{N}\sum_i c_{ii}$.*

In essence, the same comments at the end of Section 3 hold for the nonholonomic case as well.

5 Simulations

To support the results of the previous sections we provide a series of computer simulations.

The first simulation involves seven single integrator agents of the form (1) that evolve under the control law (2). The communication sets are chosen in order to satisfy the connectivity requirement for the formation graph. The interagent desired relative positions satisfy $\dot{q}^* = -\frac{1}{7}\sum_{i=1}^{N} c_{ii} = \begin{bmatrix} -0.0177 & 0.01 \end{bmatrix}^T$. Graphs I-IV of Figure 1 show the evolution in time of the multi-agent team. As can be seen in graph IV, the interagent velocities vectors are stabilized at steady state to a common value. This is also depicted in the velocity diagrams (Figure 2) in both x and y directions, which show that the agents reach the expected velocity volume imposed by Corollary 5 in both directions (-0.0177 in the x and 0.01 in the y-direction.)

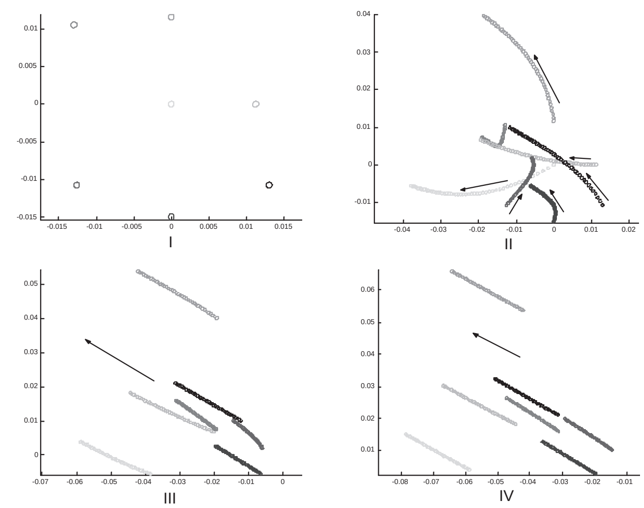

Fig. 1. Formation infeasibility results in velocity alignment.

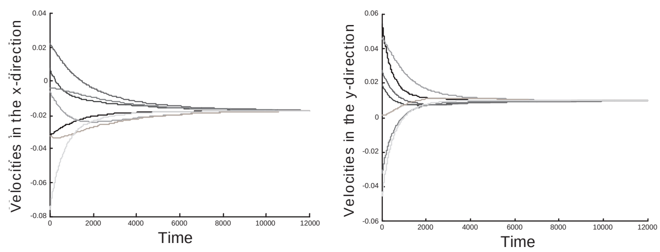

Fig. 2. Velocity diagrams for the first simulation.

The next simulation involves four nonholonomic agents and a connected formation graph. The interagent desired relative positions satisfy $\dot{q}^* = -\frac{1}{4}\sum_{i=1}^{N} c_{ii} = \begin{bmatrix} 0 & 0.02 \end{bmatrix}^T$, so that the resulting velocity vector drives the agents to the "north" direction. As can be seen in graphs I-IV of Figure 3, the nonholonomic agents are eventually stabilized to a common velocity. This velocity is equal to q^*, as depicted in the velocity diagram in Figure 4, where the velocity of the agents converges to the zero value in the x-direction and to the expected value 0.02 in the y-direction, in accordance with Corollary 8.

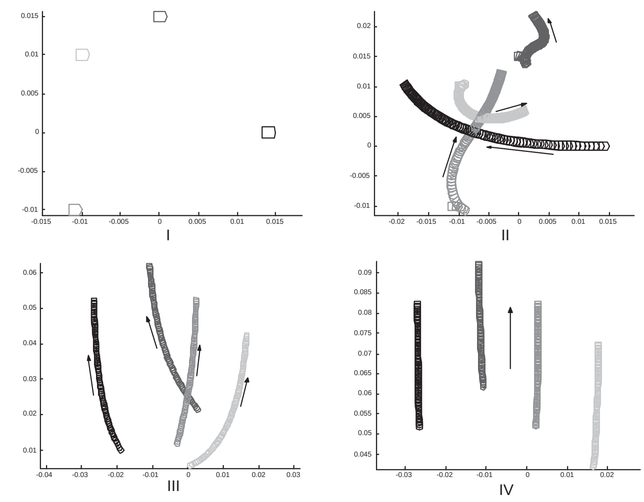

Fig. 3. Formation infeasibility results in velocity alignment for four nonholonomic agents.

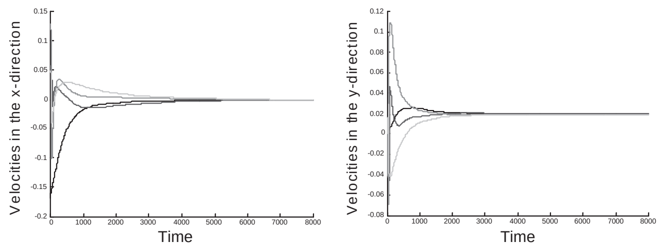

Fig. 4. Velocity diagrams for the second simulation.

6 Conclusions

We provided a connection between formation infeasibility and velocity alignment in kinematic multi-agent systems. Specifically, we showed that formation infeasibility forces the agents' velocity vectors to a common value at steady state, for both the cases of agents with single integrator and nonholonomic unicycle-type kinematics. An analytic expression of the common velocity vector was also obtained. The results were supported through computer simulations.

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