Connectedness Preserving Distributed Swarm Aggregation for Multiple Kinematic Robots

Dimos V. Dimarogonas and Kostas J. Kyriakopoulos, *Member, IEEE*

***Abstract*— A distributed swarm aggregation algorithm is developed for a team of multiple kinematic agents. Specifically, each agent is assigned with a control law which is the sum of two elements: a repulsive potential field, which is responsible for the collision avoidance objective, and an attractive potential field, that forces the agents to converge to a configuration where they are close to each other. Furthermore, the attractive potential field forces the agents that are initially located within the sensing radius of an agent to remain within this area for all time. In this way, the connectivity properties of the initially formed communication graph are rendered invariant for the trajectories of the closed-loop system. It is shown that under the proposed control law agents converge to a configuration where each agent is located at a bounded distance from each of its neighbors. The results are also extended to the case of nonholonomic kinematic unicycle-type agents and to the case of dynamic edge addition. In the latter case, we derive a smaller bound in the swarm size than in the static case.**

***Index Terms*— Multi-agent coordination; Distributed swarm coordination; Graph connectivity; Dynamic Graphs.**

I. Introduction

Navigation of multi-agent systems is a field that has recently gained increasing attention both in the robotics and the control communities. While most efforts in the past focused on centralized planning [21], specific real-world applications have lead researchers throughout the globe to turn their attention to decentralized concepts. This work is motivated from the field of micro robotics [15], where a large number of autonomous micro robots must cooperate in the sub micron level. Other applications include decentralized air traffic management systems [37], distributed control of multiple UAV's [33] and coordination of multiple robots in hazardous civil operations.

The variations of the approaches so far lie in the specifications that the control design should impose on the multi-agent team, e.g., formation convergence and achievement of flocking behavior. In the formation control case, agents must converge to a desired configuration encoded by the their relative positions. Many control schemes that achieve formation stabilization in a distributed manner have been proposed, e.g., [1], [27],[14],[22],[20], [3],[10],[7],[19]. The agreement problem, where agents must converge to the same point in the state space [26],[29],[5],[17],[32], is also relevant. On the other hand, flocking behavior involves convergence of the velocity vectors and orientations of the agents to a common value at steady state ([16], [36],[28]). In many cases, the collision avoidance objective was not taken into account. It is obvious that this specification is necessary for the implementation of such algorithms in robotic systems. Collision avoidance has been dealt with in [36],[28], [23],[12],[24],[31].

The objective of this paper is distributed swarm aggregation with collision avoidance. Each agent is assigned a control law which is the sum of two elements: a repulsive potential field, which is responsible for the collision avoidance objective, and an attractive potential field, that forces the agents to converge to a configuration where they are close to each other. Furthermore, the attractive potential field forces the agents that are initially located within the sensing zone of an agent to remain within this zone for all time. Hence the control design renders the set of edges of the initially formed communication graph positively invariant for the trajectories of the closed loop system. In this way, if the communication graph, which is formed based on the initial relative distances between the team members, is connected, then it remains connected throughout the closed loop system evolution.

A centralized version of this model was analyzed in [24],[12],[13]. The innovation of our approach with respect to the aforementioned, is the fact that the control design is distributed. The collision avoidance objective is guaranteed through the use of repulsive potentials that disappear whenever agents are outside the sensing zone of one another, respecting the agents' limited sensing capabilities. Thus, the need of all to all communication for collision avoidance [36] is no longer needed. The framework also takes into account nonholonomic constraints. We also provide a control law that renders the connectivity properties of the initially formed communication graph invariant for the trajectories of the closed loop system, and treat the dynamic edge addition case as well. In the latter case, it is shown that the resulting swarm size is smaller than that of the static graph case treated previously. Connectivity preserving algorithms for single-integrator agents have recently been dealt with in [17],[18], [38],[39], while [9] treats the case of nonholonomic agents. In contrast to [17],[18],[9] that treat the agreement problem, the control law of this paper considers the collision avoidance objective and moreover it is a distributed control law, contrary to the centralized approach of [38],[39].

In summary, the innovations and contributions of the paper

Dimos V. Dimarogonas is with KTH ACCESS Linnaeus Center, School of Electrical Eng., Royal Institute of Technology (KTH), SE-100 44,Stockholm, Sweden, email: dimos@ee.kth.se. Kostas J. Kyriakopoulos is with the Control Systems Lab., Mechanical Eng. Dept., National Technical University of Athens, 9 Heroon Polytechniou Street, Zografou 15780, Greece,email: kkyria@central.ntua.gr. The authors were supported by EU through contract I-SWARM (IST-2004-507006). The first author'work was partially done within TAIS-AURES program (297316-LB704859), funded by the Swedish Governmental Agency for Innovation Systems (VINNOVA) and the Swedish Defence Materiel Administration (FMV). He was also partially supported by the Swedish Research Council, the Swedish Foundation for Strategic Research, and EU NoE HYCON.

are (i) the use of distributed control laws for swarm aggregation, (ii) the inclusion of the collision avoidance objective in the connectivity preserving control law, (iii) the application of the results to nonholonomic agents, (iv) the extension of the results to dynamic edge addition and most importantly (v) the fact that we prove that should a dynamic graph formulation be applied, the resulting swarm size is smaller than that of the static graph case. A preliminary conference version of the paper appeared in [8], compared to which we provide in the current paper a complete analysis of the nonholonomic and dynamic graph cases, as well as a more detailed simulations' section.

The rest of the paper is organized as follows: Section II describes the system and states the problems treated in this paper. Section III presents the proposed control strategy for the single integrator case. The stability analysis of the control strategy is included in Section IV. Section V extends the results to the case of unicycle-type kinematic robots. In Section VI we reformulate the problem to allow for dynamic edge addition and provide an improved result on the swarm size than the one provided in Section IV. Computer simulation results are included in section VII while section VIII provides a summary of the results of this paper.

II. System and Problem Definition

Consider N (point) agents operating in $W \subset \mathbb{R}^2$. We denote the position of agent i by $q_i \in \mathbb{R}^2$. The configuration space is spanned by $q = [q_1^T, \ldots, q_N^T]^T$. The motion of each agent is described by the single integrator kinematic model:

$$\dot{q}_i = u_i, i \in \mathcal{N} = \{1, \ldots, N\} \tag{1}$$

where $u_i \in \mathbb{R}^2$ denotes the velocity (control input) for each agent.

For the objective of swarm aggregation, each agent i is assigned to a specific subset N_i of the rest of the team, called agent i's *communication set*, that includes the agents with which it can communicate in order to achieve the desired aggregation objective. Inter-agent communication can be encoded in terms of a *communication graph*:

Definition 1: The *communication graph* $G = (V, E)$ is an undirected graph that consists of a set of vertices $V = \{1, ..., N\}$ indexed by the team members, and a set of edges, $E = \{(i, j) \in V \times V | i \in N_j\}$ containing pairs of nodes that represent inter-agent communication specifications.

The definition of the set N_i is provided later. Apart from the aggregation objective, it is required that the agents do not collide. Collision avoidance is meant in the sense that the point agents are not simultaneously found at the same points. The collision avoidance procedure is distributed in the sense that each agent has to have only local knowledge of the agents that are very close at *each time instant*. Since agent i can sense agents located at a distance no larger than d at each time instant, we assume that for the collision avoidance objective, agent i has knowledge of the positions of agents located at a distance no larger than a radius d_1, where $0 < d_1 \leq d$, at each time instant. The subset of $\mathcal{N}$ including the agents that are located at a distance no larger than a radius d_1 from agent i is denoted by M_i. Hence

$$M_i = \{j \in \mathcal{N}, j \neq i : \|q_i - q_j\| \leq d_1\} \tag{2}$$

While M_i contains the agents located at a distance no larger than d_1 from agent i at each time instant, the communication set N_i is defined in a slightly different manner in relation with the proposed control design. More specifically, we show in the following section that the proposed control law forces the agents that are initially located within the sensing zone of an agent to remain within this area for all time. In this way, no edges are lost and if the communication graph is initially connected, then it remains connected for all time. Therefore the set N_i is defined as the set that agent i can sense when it is located at its initial position, $q_i(0)$:

$$N_i = \{j \in \mathcal{N}, j \neq i : \|q_i(0) - q_j(0)\| < d\}. \tag{3}$$

Let $G = (V, E)$ denote the initially formed communication graph under the ruling (3), according to Definition 1. An edge between agents i, j exists if they are initially located within distance d from each other, i.e. $(i, j) \in E \Leftrightarrow j \in N_i$ if and only if $\|q_i(0) - q_j(0)\| < d$. By showing that for all pairs of agents (i, j) s.t. $\|q_i(0) - q_j(0)\| < d$ the proposed controller guarantees that $\|q_i(t) - q_j(t)\| < d$ for all $t > 0$, the edges are guaranteed to remain invariant (i.e. agents i, j remain within distance d from one another) and hence the communication graph itself, remains invariant throughout the closed loop system evolution. This result is stated and proved in Lemma 3 of the paper. The case of dynamic edge addition will be considered in Section VI. On the other hand, the set M_i changes at time instances when an agent $j \neq i$ enters or leaves the set $\{q : \|q_i - q\| \leq d_1\}$. Therefore the (distributed) control law is of the form $u_i = u_i(q_i, q_j), j \in N_i \cup M_i$.

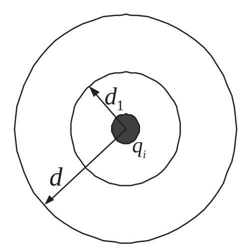

Fig. 1. Each agent has sensing radius d. For the collision avoidance objective, it requires knowledge of the positions of agents at distance less than $d_1 < d$ at each time instant.

III. Control Strategy

We first define a repulsive potential field $V_{ij} : \mathbb{R}^2 \to \mathbb{R}_+$ to deal with the collision avoidance specification between agents i and $j \in M_i$. We consider both the cases of a bounded and an unbounded repulsive potential. V_{ij} is required to possess the following properties:

1) V_{ij} is a function of the square norm of the distance between agents i, j, i.e.

$$V_{ij} = V_{ij}\left(\underbrace{\|q_i - q_j\|^2}_{\beta_{ij}}\right) = V_{ij}(\beta_{ij})$$

2) V_{ij} attains its maximum value whenever $\beta_{ij} \to 0$. This maximum value is finite when the potential force is bounded. For the case of an unbounded repulsive potential, we require that $V_{ij} \to \infty$ whenever $\beta_{ij} \to 0$.
3) It is everywhere continuously differentiable.
4) $\frac{\partial V_{ij}}{\partial q_i} = 0$ and $V_{ij} = 0$ whenever $\beta_{ij} > d_1^2$.
5) The partial derivative $\rho_{ij} \triangleq \frac{\partial V_{ij}}{\partial \beta_{ij}}$ satisfies $\rho_{ij} < 0$ for $0 < \beta_{ij} < d_1^2$ and $\rho_{ij} = 0$ for $\beta_{ij} \geq d_1^2$.

It is straightforward to see that if the potential field satisfies these requirements, then agent i needs to have only knowledge of the states of agents within M_i at each time instant to fulfil the collision avoidance objective. The fourth requirement also guarantees that $\sum\limits_{j\in M_i} \frac{\partial V_{ij}}{\partial q_i} = \sum\limits_{j\neq i} \frac{\partial V_{ij}}{\partial q_i}$. The gradient with respect to q and the partial derivative of V_{ij} with respect to q_i are computed by $\nabla V_{ij} = 2\rho_{ij} D_{ij} q$ and $\frac{\partial V_{ij}}{\partial q_i} = 2\rho_{ij}\left(D_{ij}\right)_i q$ where the matrices D_{ij},$(D_{ij})_i$, for $i < j$ are given by $D_{ij} = \tilde{D}_{ij} \otimes I_2$, where $\left(\tilde{D}_{ij}\right)_{ii} = \left(\tilde{D}_{ij}\right)_{jj} = 1$, $\left(\tilde{D}_{ij}\right)_{ij} = \left(\tilde{D}_{ij}\right)_{ji} = -1$ and $\left(\tilde{D}_{ij}\right)_{kl} = 0$ for $k, l \neq i, j$, and $\left(D_{ij}\right)_i = \begin{bmatrix} O_{1\times(i-1)} & 1 & O_{1\times(j-i-1)} & -1 & O_{1\times(N-j)} \end{bmatrix} \otimes I_2$. The definition of $D_{ij}, (D_{ij})_i$ for $i > j$ is straightforward. This definition of V_{ij} guarantees that the potential field has the following important symmetry property: $\rho_{ij} = \rho_{ji}, \forall i, j \in \mathcal{N}, i \neq j$.

For the purpose of aggregation, we define an attractive potential $W_{ij} : \mathbb{R}^2 \to \mathbb{R}_+$ between agents i and $j \in N_i$, which is required to have the following properties:

1) W_{ij} is a function of the square norm of the distance between agents i, j, i.e.
$$W_{ij} = W_{ij}\left(\|q_i - q_j\|^2\right) = W_{ij}\left(\beta_{ij}\right)$$
2) W_{ij} is defined on $\beta_{ij} \in [0, d^2)$.
3) $W_{ij} \to \infty$ whenever $\beta_{ij} \to d^2$.
4) It is everywhere continuously differentiable for $\beta_{ij} \in [0, d^2)$.
5) The partial derivative $p_{ij} \triangleq \frac{\partial W_{ij}}{\partial \beta_{ij}}$ satisfies $p_{ij} > 0$ for $0 \leq \beta_{ij} < d^2$.

Function W_{ij} is hence defined to ensure that agents that are located at a distance no larger than d from agent i at time $t = 0$, remain within agent i's sensing zone for all $t > 0$. We have $\nabla W_{ij} = 2p_{ij} D_{ij} q$ and $\frac{\partial W_{ij}}{\partial q_i} = 2p_{ij}\left(D_{ij}\right)_i q$ where $p_{ij} \triangleq \frac{\partial W_{ij}}{\partial \beta_{ij}}$ and the matrices D_{ij},$(D_{ij})_i$ were defined previously. The following symmetry property holds in this case as well: $p_{ij} = p_{ji},\ \forall j \in N_i$.

The proposed control law for each agent i is given as the sum of the negative gradients of the two potentials in the q_i direction:
$$u_i = -\sum_{j\in N_i} \frac{\partial W_{ij}}{\partial q_i} - \sum_{j\in M_i} \frac{\partial V_{ij}}{\partial q_i} \tag{4}$$
The control law can also be written as $u_i = -2\sum\limits_{j\in N_i} p_{ij}\left(q_i - q_j\right) - 2\sum\limits_{j\in M_i} \rho_{ij}\left(q_i - q_j\right)$. Since the proposed control law of i requires knowledge only of the states of agents belonging to $N_i \cup M_i$, it respects the sensing limitations of each agent. It is hence clearly a *distributed* control design.

IV. Stability Analysis

The function $V = \sum\limits_i \left(\sum\limits_{j\in N_i} W_{ij} + \sum\limits_{j\neq i} V_{ij} \right)$ is used as a candidate Lyapunov function for the multi-agent system. Differentiating V with respect time we get $\dot{V} = (\nabla V)^T \cdot \dot{q}$. We first compute the gradient of V. We have
$$\sum_i \sum_{j\in N_i} \nabla W_{ij} = 2\left(\sum_i \sum_{j\in N_i} p_{ij} D_{ij}\right) q = 4\left(P \otimes I_2\right) q$$
where the $N \times N$ matrix P can be shown to be given by $P_{ii} = \sum\limits_{j\in N_i} p_{ij}$, $P_{ij} = -p_{ij}$ for $j \in N_i, i \neq j$, and $P_{ij} = 0$ for $j \notin N_i$. The form of matrix P was derived based on the form of the D_{ij} matrices.

We can also compute
$$\sum_i \sum_{j\neq i} \nabla V_{ij} = 2\left(\sum_i \sum_{j\neq i} \rho_{ij} D_{ij}\right) q = 2\left(R_1 \otimes I_2\right) q$$
where matrix R_1 can be computed by
$$\left(R_1\right)_{ij} = \begin{cases} \sum\limits_{j\neq i} \rho_{ij} + \sum\limits_{j\neq i} \rho_{ji}, i = j \\ -\rho_{ij} - \rho_{ji}, i \neq j \end{cases}$$
The gradient of V is now given by $\nabla V = 4\left(P \otimes I_2\right) q + 2\left(R_1 \otimes I_2\right) q$. The time derivative of the stack vector of the agents' positions is given by
$$\dot{q} = \begin{bmatrix} -\sum\limits_{j\in N_1} \frac{\partial W_{1j}}{\partial q_1}^T, & \cdots & , -\sum\limits_{j\in N_N} \frac{\partial W_{Nj}}{\partial q_N}^T \end{bmatrix}^T + \begin{bmatrix} -\sum\limits_{j\in M_1} \frac{\partial V_{1j}}{\partial q_1}^T, & \cdots & , -\sum\limits_{j\in M_N} \frac{\partial V_{Nj}}{\partial q_N}^T \end{bmatrix}^T$$
The first term on the right hand side of the previous equation is given by
$$\begin{bmatrix} -\sum\limits_{j\in N_1} \frac{\partial W_{1j}}{\partial q_1}^T, & \cdots & , -\sum\limits_{j\in N_N} \frac{\partial W_{Nj}}{\partial q_N}^T \end{bmatrix}^T = -2\left(P \otimes I_2\right) q$$
Note also that
$$\begin{bmatrix} -\sum\limits_{j\in M_1} \frac{\partial V_{1j}}{\partial q_1}^T, & \cdots & , -\sum\limits_{j\in M_N} \frac{\partial V_{Nj}}{\partial q_N}^T \end{bmatrix}^T = \begin{bmatrix} -\sum\limits_{j\neq 1} \frac{\partial V_{1j}}{\partial q_1}^T, & \cdots & , -\sum\limits_{j\neq N} \frac{\partial V_{Nj}}{\partial q_N}^T \end{bmatrix}^T = -2\left(R \otimes I_2\right) q$$
The elements of the matrix R are computed based on the form of the D_{ij} matrix and are given by $R_{ii} = \sum\limits_{j\neq i} \rho_{ij}$ and $R_{ij} = -\rho_{ij}$, for $i \neq j$. Hence $\dot{q} = -2\left(P \otimes I_2\right) q - 2\left(R \otimes I_2\right) q$. Using now the symmetry of the potentials we get $\rho_{ij} = \rho_{ji} \Rightarrow R_1 = 2R$, so that $\dot{V} = (\nabla V)^T \cdot \dot{q} = -\left(4\left(P \otimes I_2\right) q + 2\left(R_1 \otimes I_2\right) q\right)^T \cdot \left(2\left(P \otimes I_2\right) q + 2\left(R \otimes I_2\right) q\right)$, and since $R_1 = 2R$,
$$\Rightarrow \dot{V} = -8\left\|\left(\left(P \otimes I_2\right) q + \left(R \otimes I_2\right) q\right)\right\|^2 \leq 0 \tag{5}$$
We now state the first result of this paper:

Theorem 1: Assume that the swarm (1) evolves under the control law (4). Then the system reaches a configuration in which $u = 0$, i.e. $u_i = 0$ for all $i \in \mathcal{N}$.

Proof: The level sets of V are compact and invariant with respect to the relative positions of adjacent agents. Specifically, the set $\Omega_c = \{q : V(q) \leq c\}$ for $c > 0$ is closed by the continuity of V. For all $(i,j) \in E$ we have $V \leq c \Rightarrow W_{ij} \leq c \Rightarrow \|q_i - q_j\| \leq \sqrt{W_{ij}^{-1}(c)}$. Equation (5) and LaSalle's invariance principle guarantee that the system converges to the largest invariant subset of the set $S = \{q : ((P + R) \otimes I_2)\, q = 0\}$. Since $u = \dot{q} = -2\,(P \otimes I_2)\, q - 2\,(R \otimes I_2)\, q$, we have $u = 0$ and the result follows. $\diamondsuit$

The next Lemma establishes collision avoidance in the case of an unbounded potential:

Lemma 2: Consider the system (1) driven by the control law (4) and starting from a feasible set of initial conditions $\mathcal{I}(q) = \{q |\, \|q_i - q_j\| > 0, \forall i, j \in \mathcal{N}, i \neq j\}$. Assume that the repulsive potential is unbounded. Then $\mathcal{I}(q)$ is invariant for the trajectories of the closed-loop system.

Proof: For every initial condition $q(0) \in \mathcal{I}(q)$, the time derivative of V remains non-positive for all $t \geq 0$, by virtue of (5). Hence $V(q(t)) \leq V(q(0)) < \infty$ for all $t \geq 0$. Since $V \to \infty$ when $\|q_i - q_j\| \to 0$ for at least one pair $i, j \in \mathcal{N}$, we conclude that $q(t) \in \mathcal{I}(q)$, for all $t \geq 0$. $\diamondsuit$

The next result of the paper involves the fact that the proposed control law forces agents that are initially located within distance d from each other to remain within this distance for all time. Hence the definition of N_i is rendered meaningful since each agent i does not have to violate its sensing constraints in order to sense agents within N_i as the closed loop system evolves. In other words, the control design also guarantees that an agent j initially located at a distance less than d from i, will never leave the sensing zone of i. This is proved in the following Lemma:

Lemma 3: Consider the multi-agent system (1) driven by the control law (4). The set $\mathcal{J}(q) = \{q |\, \|q_i - q_j\| < d, \forall\,(i,j) \in E\}$ is invariant for the trajectories of the closed loop system.

Proof: Since $V(q(t)) \leq V(q(0)) < \infty$ for all $t \geq 0$ and $V \to \infty$ when $\|q_i - q_j\| \to d$ for at least one pair $(i,j) \in E$, we conclude that $q(t) \in \mathcal{J}(q)$, for all $t \geq 0$. $\diamondsuit$

Based on the fact that all agents initially located within distance d from each other remain within this distance for all time, the set N_i is a static set. Hence no new edges are created even when an agent not initially located within the sensing radius of another, enters inside this set at some time instant $t > 0$. The case of dynamic edge addition, i.e. adding new edges to the communication graph each time a new agent enters the sensing zone of another, will be treated in Section VI. In essence, starting from the set $\mathcal{J}(q) \cap \mathcal{I}(q)$, the communication graph remains invariant (no edges are lost) and collisions are avoided.

In the sequel, we derive bounds on the swarm size. We first show that the "swarm center" $\bar{q} \triangleq \frac{1}{N}\sum\limits_{i=1}^{N} q_i$ remains constant, i.e. $\bar{q}(t) = \bar{q}(0)$ for all $t \geq 0$. This is proven by the fact that $\dot{\bar{q}} = \frac{1}{N}\sum\limits_{i=1}^{N} \dot{q}_i = -\frac{2}{N}\sum\limits_{i=1}^{N}\big(\sum\limits_{j \in M_i} \rho_{ij}\,(q_i - q_j) + \sum\limits_{j \in N_i} p_{ij}\,(q_i - q_j)\big) = 0$. Since $\bar{q}$ is constant, we assume without loss of generality that it is the origin of the coordinate system, i.e. $\bar{q} = 0$.

Moreover, at an equilibrium point we have $u = 0$, by virtue of Theorem 1. Considering the function $\Phi = \frac{1}{2}\sum\limits_i q_i^T q_i$ and taking its time derivative we have $\Phi = \frac{1}{2}\sum\limits_i q_i^T q_i \Rightarrow \dot{\Phi} = \sum\limits_i q_i^T \dot{q}_i = 0$. Hence, we can derive a conclusion similar to the one in [13]:

$$\begin{aligned}\dot{\Phi} &= -2\sum_i \left(q_i^T \left(\sum_{j \in M_i} \rho_{ij}\,(q_i - q_j) + \sum_{j \in N_i} p_{ij}\,(q_i - q_j) \right) \right) \\ &= -\sum_i \left(\sum_{j \in M_i} \rho_{ij}\,\|q_i - q_j\|^2 + \sum_{j \in N_i} p_{ij}\,\|q_i - q_j\|^2 \right) = 0\end{aligned}$$

and hence at an equilibrium position:

$$\sum_i \sum_{j \in N_i} p_{ij}\,\|q_i - q_j\|^2 = \sum_i \sum_{j \in M_i} |\rho_{ij}|\,\|q_i - q_j\|^2 \qquad (6)$$

since $\rho_{ij} \leq 0, \forall j \in M_i$. The last equation enables us to derive bounds on various distances that describe the swarm size. These are based on the bounds on the designed potential functions. The attractive potential is chosen so that $p_{ij} \geq a$ where $a > 0$. An example is given by the function $W_{ij}(\beta_{ij}) = \frac{b}{d^2 - \beta_{ij}}$, for $\mu < \beta_{ij} < d^2$ and $W_{ij}(\beta_{ij}) = a\beta_{ij}$, for $0 \leq \beta_{ij} \leq \mu$, where the positive parameters b, μ are chosen in order to render the function W_{ij} everywhere continuously differentiable. We then have $\sum\limits_i \sum\limits_{j \in N_i} p_{ij}\,\|q_i - q_j\|^2 \geq a\sum\limits_i \sum\limits_{j \in N_i} \|q_i - q_j\|^2$.

For the repulsive potential, we consider the cases of both unbounded and bounded repulsion forces. In the first case, we can design V_{ij} so that ρ_{ij} satisfies the bound: $|\rho_{ij}| \leq \frac{\rho}{\beta_{ij}}$, where $\rho > 0$. An example of such a potential is given by $V_{ij}(\beta_{ij}) = \rho \ln\left(\frac{1}{\beta_{ij}}\right)$, for $\beta_{ij} < c$, $V_{ij}(\beta_{ij}) = h\left(\beta_{ij} - d^2\right)^2$, for $c \leq \beta_{ij} < d^2$ and $V_{ij}(\beta_{ij}) = 0$, for $\beta_{ij} \geq d^2$, where the positive parameters c, h are chosen in order to render the function V_{ij} everywhere continuously differentiable. We then have $\sum\limits_i \sum\limits_{j \in M_i} |\rho_{ij}|\,\|q_i - q_j\|^2 \leq \rho\sum\limits_i |M_i|$, where $|M_i|$ is the cardinality of M_i. Equation (6) yields $\sum\limits_i \sum\limits_{j \in N_i} \|q_i - q_j\|^2 = \sum\limits_i \sum\limits_{j \in N_i} \beta_{ij} \leq \frac{\rho}{a}\sum\limits_i |M_i|$. The right hand side is maximized whenever each agent is located at a distance less than d_1 from all other agents, i.e. the repulsive potential is active for all pairs $i, j \in \mathcal{N}$. We then have $\sum\limits_i |M_i| \leq N\,(N-1)$. For each pair of agents that form an edge, an ultimate bound is then given by:

$$\beta_{ij} \leq \frac{\rho}{a} N\,(N-1)\,,\ \forall\,(i,j) \in E \qquad (7)$$

We then have:

Theorem 4: Assume that the swarm (1) evolves under the control law (4) and the initially formed communication graph is connected. Denote by $\beta_{\max}$ the maximum distance between two members of the group, i.e. $\beta_{\max} = \max\limits_{i,j \in \mathcal{N}} \|q_i - q_j\|^2$.

Under the preceding assumptions, the following bound holds at steady state: $\beta_{\max} \leq \frac{\rho}{a} N\left(N-1\right)^2$.

Proof: Since the graph is connected, the maximum length of a path connecting two arbitrary vertices is $N-1$. The result now follows from (7). $\diamondsuit$

In the case of bounded repulsion, the repulsive potential can be constructed so that ρ_{ij} satisfies the bound: $\left|\rho_{ij}\right| \leq \frac{\sigma}{\sqrt{\beta_{ij}}}$, where $\sigma > 0$. We then have $\sum\limits_i \sum\limits_{j\in M_i} \left|\rho_{ij}\right| \left\|q_i - q_j\right\|^2 \leq \sum\limits_i \sum\limits_{j\in M_i} \sigma \left\|q_i - q_j\right\| \leq \sigma d_1 \sum\limits_i \left|M_i\right| \leq \sigma d_1 N\left(N-1\right)$ since $\beta_{ij} \leq d_1^2$ for all $j \in M_i$. Using again equation (7), a better bound on the maximum distance of agents forming an edge can be achieved: $\beta_{ij} \leq \frac{\sigma}{a} d_1 N\left(N-1\right),\ \forall\left(i,j\right)\in E$. A result similar to that of Theorem 4 holds. The use of the parameter d_1 provides additional freedom to the control designer in choosing the design parameters. Note however that in this case, collision avoidance is no longer guaranteed.

V. The case of nonholonomic kinematic unicycle-type agents

In this section, we consider the case of a swarm of multiple unicycles. The stability analysis requires tools from nonsmooth analysis, a review of which is given in the next subsection.

A. Tools from Nonsmooth Analysis

Definition 2: [11] For a finite dimensional state-space, the vector function $x(.)$ is called a *Filippov solution* of $\dot{x} = f(x)$, where f is measurable and essentially locally bounded, if it is absolutely continuous and $\dot{x} \in K[f](x)$ almost everywhere where $K[f](x) \equiv \overline{co}\{\lim_{x_i\to x} f(x_i)|x_i \notin N_0\}$ and N_0 is a set of measure zero that contains the set of points where f is not differentiable.

Theorem 5: [34] Let x be a Filippov solution to $\dot{x} = f(x)$ on an interval containing t and $V: \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz and regular function. Then $V(x(t))$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and $\frac{d}{dt}V(x(t)) \in^{a.e.} \dot{\tilde{V}}(x) := \bigcap_{\xi\in\partial V(x(t))} \xi^T K[f](x(t))$, where "a.e." stands for "almost everywhere" and ∂V is *Clarke's generalized gradient* [4].

The Lyapunov function V we use here is smooth and hence regular, and thus $\partial V(x) = \{\nabla V(x)\}\ \forall x$. We will use the following nonsmooth version of LaSalle's invariance principle:

Theorem 6: [34] Let Ω be a compact set such that every Filippov solution to $\dot{x} = f(x), x(0) = x(t_0)$ starting in Ω is unique and remains in Ω for all $t \geq t_0$. Let $V: \Omega \to \mathbb{R}$ be a time independent regular function such that $v \leq 0, \forall v \in \dot{\tilde{V}}$ (if $\dot{\tilde{V}} = \emptyset$ this is trivially satisfied). Define $S = \{x \in \Omega | 0 \in \dot{\tilde{V}}\}$. Then every trajectory in Ω converges to the largest invariant set in the closure of S.

B. Control Design and Stability Analysis

In this section, the proposed control law is presented. Consider N nonholonomic agents operating in $W \subset \mathbb{R}^2$. Let $q_i = [x_i, y_i]^T \in \mathbb{R}^2$ denote the position of agent i. The configuration space is spanned by $q = [q_1^T, \ldots, q_N^T]^T$. Each agent $i \in \mathcal{N}$ has an orientation θ_i with respect to the global coordinate frame. The configuration of each agent is represented by $p_i = \left[\begin{array}{cc} q_i^T & \theta_i \end{array}\right] \in \mathbb{R}^2 \times (-\pi, \pi]$. The motion of the agents is described by the following nonholonomic kinematics:

$$\begin{array}{l} \dot{x}_i = u_i \cos\theta_i \\ \dot{y}_i = u_i \sin\theta_i \\ \dot{\theta}_i = \omega_i \end{array}, i \in \mathcal{N} = \{1, \ldots, N\} \tag{8}$$

where u_i, ω_i denote the translational and rotational velocity of agent i, respectively. Similarly to the previous case, the aggregation control law for each unicycle is of the form $u_i = u_i\left(p_i, p_j\right), \omega_i = \omega_i\left(p_i, p_j\right), j \in N_i \cup M_i, i \in \mathcal{N}$. Consider again $V = \sum\limits_i \left(\sum\limits_{j\in N_i} W_{ij} + \sum\limits_{j\neq i} V_{ij}\right)$ as a candidate Lyapunov function. Since the proposed control law will be discontinuous we will use Theorem 5 for the time derivative of V. Since V is smooth we have $\partial V = \{\nabla V\}$, which is calculated as $\nabla V = 4\left(P\otimes I_2\right)q + 2\left(R_1\otimes I_2\right)q = 4\left(\left(P\otimes I_2\right)q + \left(R\otimes I_2\right)q\right)$. We define $P + R \triangleq F$. We have $q = [q_1^T, \ldots, q_N^T]^T = [[x_1, y_1]^T, \ldots, [x_N, y_N]^T]^T$ and we let x,y denote the stack vectors of the agents coefficients in the x,y coordinates respectively, i.e., $x = [x_1, \ldots, x_N]^T$ and $y = [y_1, \ldots, y_N]^T$. Furthermore, let $(a)_i$ denote the i-th element of the vector a. Then:

Theorem 7: Assume that the nonholonomic swarm (8) evolves under the control law

$$u_i = -\mathrm{sgn}\left\{f_{xi}\cos\theta_i + f_{yi}\sin\theta_i\right\} \cdot \left(f_{xi}^2 + f_{yi}^2\right)^{1/2} \tag{9}$$

$$\omega_i = -\left(\theta_i - \arctan 2\left(f_{yi}, f_{xi}\right)\right) \tag{10}$$

where $\left(Fx\right)_i = f_{xi}, \left(Fy\right)_i = f_{yi}$. Then the system reaches the equilibrium points of the single integrator case, i.e. a configuration in which $\left(\left(P\otimes I_2\right) + \left(R\otimes I_2\right)\right)q = 0$.

Proof: The generalized time derivative of V is calculated by

$$\begin{array}{l} \dot{\tilde{V}} = \left(\nabla V\right)^T \\ \cdot K\left[\begin{array}{ccccc} u_1\cos\theta_1, & u_1\sin\theta_1, & \ldots & u_N\cos\theta_N, & u_N\sin\theta_N \end{array}\right]^T \\ \subset 4\left(\left(P\otimes I_2\right)q + \left(R\otimes I_2\right)q\right)^T \\ \cdot K\left[\begin{array}{ccccc} u_1\cos\theta_1, & u_1\sin\theta_1, & \ldots & u_N\cos\theta_N, & u_N\sin\theta_N \end{array}\right]^T \\ \subset 4\left(Fx\right)^T\left[\begin{array}{ccc} K\left[u_1\right]\cos\theta_1, & \ldots & K\left[u_N\right]\cos\theta_N \end{array}\right]^T \\ +4\left(Fy\right)^T\left[\begin{array}{ccc} K\left[u_1\right]\sin\theta_1, & \ldots & K\left[u_N\right]\sin\theta_N \end{array}\right]^T \\ \subset \sum\limits_{i\in\mathcal{N}} \left\{4K\left[u_i\right]\left(\left(Fx\right)_i\cos\theta_i + \left(Fy\right)_i\sin\theta_i\right)\right\} \end{array}$$

where we used Theorem 1.3 in [30] to calculate the inclusions of the Filippov set. Since $K\left[\mathrm{sgn}(x)\right]x = \{|x|\}$ ([30],Theorem 1.7), the choice of control laws (9),(10) results in $\dot{V} = -\sum\limits_i \left\{4\left|f_{xi}\cos\theta_i + f_{yi}\sin\theta_i\right|\left(f_{xi}^2 + f_{yi}^2\right)^{1/2}\right\} \leq 0$, so that the generalized derivative of V reduces to a singleton. By Theorem 6, the agents converge to the largest invariant subset of the set $S = \{\left(f_{xi} = f_{yi} = 0\right) \vee \left(f_{xi}\cos\theta_i + f_{yi}\sin\theta_i = 0\right), \forall i \in \mathcal{N}\}$. However, for each $i \in \mathcal{N}$, we have $\left|\omega_i\right| = \frac{\pi}{2}$ whenever $f_{xi}\cos\theta_i + f_{yi}\sin\theta_i = 0$. In particular [35], this choice of ω_i renders the surface $f_{xi}\cos\theta_i + f_{yi}\sin\theta_i = 0$ non-invariant for agent i, whenever i is not located at $f_{xi} = f_{yi} = 0$. Hence the largest invariant set S_0 contained in S is $S \supset S_0 = \{f_{xi} = f_{yi} = 0, \forall i \in N\}$ which is equivalent to the equilibria of the single integrator case: $((P \otimes I_2) + (R \otimes I_2))\, q = 0.\diamondsuit$

Hence the control design (9),(10) forces the nonholonomic swarm to behave in exactly the same way as in the single integrator case. This nonholonomic control strategy is an extension of the result of [35] (which is itself an extension of the earlier results [6] and [2]) for the single agent case to the case of multiple agents. The difference lies in the fact that the potential of each agent involves its relative positions with respect to neighboring agents and not its distance from a single equilibrium point.

VI. The Case of Dynamic Graphs

The previous sections involved the case where the communication graph considered was static, i.e. no new edges were added whenever an agent, not initially located within the sensing zone of another, entered its sensing zone. In practical situations however, it is more convenient to consider creation of new edges whenever an agent enters the sensing zone of another. This naturally leads to a smaller swarm size and corresponds to a more realistic formulation of the problem in hand. In this section, we consider the dynamic graph formulation in the single integrator case. The results can be also applied to the nonholonomic case of the previous section.

In this section, we consider two types of communication sets for each agent i at *each time instant*. The first one corresponds to the sensing zone of i, i.e. to the agents that agent i senses at each time instant:

$$N_i(t) = \{j \in \mathcal{N}, j \neq i : \|q_i(t) - q_j(t)\| < d\} \tag{11}$$

In order to add new communication links, we assume that a new communication link is created each time a new agent enters a subset of the sensing zone of i at some time instant. In particular, we define the set:

$$N_i^*(t) = \{j \in \mathcal{N}, j \neq i : \|q_i(t) - q_j(t)\| \leq d - \varepsilon\}$$

where $\varepsilon > 0$ a small positive scalar. It is obvious that $N_i^*(t) \subseteq N_i(t)$. We assume that the communication graph is initially formed based on the communication sets $N_i(0)$, i.e.,

$$E(0) = \{(i,j) : j \in N_i(0)\}$$

A set of edges is updated according to the following rule:

$$E(t) = E\left(t^-\right) \cup E^*(t)$$

where

$$E^*(t) = \left\{(i,j) : \left((i,j) \notin E\left(t^-\right)\right) \wedge \left(j \in N_i^*(t)\right)\right\}$$

In other words, a new edge is added whenever an agent j, that did not form an edge with i, enters at some time instant the set $N_i^*(t)$ which is a subset of sensing zone of i. By designing the control law in such a way to force agents that come to a distance $d-\varepsilon$ between them, to remain within distance d for all time afterwards, this definition of edge addition becomes meaningful since each agent has to sense only agents within its sensing zone at each time instant.

The main difference with the formulation of the static graph case is the definition of the aggregation potential between agents i and j. Specifically, we denote the aggregation potential in the dynamic graph case between any two agents by W_{ij}^d and recalling the definition of W_{ij} in the static graph case, we define W_{ij}^d as $W_{ij}^d(\beta_{ij}) = W_{ij}(\beta_{ij})$, for $(i,j) \in E(t)$ and $W_{ij}^d(\beta_{ij}) = \tilde{W}_{ij}(\beta_{ij})$, for $(i,j) \notin E(t)$. Hence whenever two agents form an edge, their aggregation potential is identical to the aggregation potential of the static graph case. Whenever an agent j forms a new edge with an agent i, the function W_{ij}^d switches from $\tilde{W}_{ij}$ to W_{ij}. The function $\tilde{W}_{ij}$ is defined in such a way that the switch to W_{ij} is held in a sufficiently smooth manner. This is encoded in the following properties:

1) $\tilde{W}_{ij}(\beta_{ij}) = W_{ij}(\beta_{ij})$, for $\|q_i(t) - q_j(t)\| \leq d - \varepsilon$
2) $\tilde{W}_{ij}(\beta_{ij}) = \tilde{W}_{ij}\left(d^2\right) = const.$, for $\|q_i(t) - q_j(t)\| > d$
3) $W_{ij}\left((d-\varepsilon)^2\right) = \tilde{W}_{ij}\left((d-\varepsilon)^2\right)$ and $\frac{\partial W_{ij}}{\partial \beta_{ij}}\left((d-\varepsilon)^2\right) = \frac{\partial \tilde{W}_{ij}}{\partial \beta_{ij}}\left((d-\varepsilon)^2\right)$.
4) $\tilde{W}_{ij}$ is everywhere continuously differentiable.
5) $\frac{\partial \tilde{W}_{ij}}{\partial \beta_{ij}} > 0$ for $d - \varepsilon < \|q_i(t) - q_j(t)\| < d$.

The control law is now defined as

$$u_i = -\sum_{(i,j)\in E} \frac{\partial W_{ij}}{\partial q_i} - \sum_{(i,j)\notin E} \frac{\partial \tilde{W}_{ij}}{\partial q_i} - \sum_{j \in M_i} \frac{\partial V_{ij}}{\partial q_i} \tag{12}$$

This definition of W_{ij}^d and in particular, $\tilde{W}_{ij}$, allows agent i to neglect agents outside its sensing zone at each time instant. Moreover, the repulsion V_{ij} is the same as in the static graph case.

The overall system can be treated as a hybrid system in which discrete transitions occur each time a new edge is added, i.e. each time two agents not forming an edge before come to a distance closer than $d-\varepsilon$ from one another. The convergence analysis is now held using the common Lyapunov function tool from hybrid stability Theory [25]. In particular, the function $V = \sum\limits_i \sum\limits_{j \neq i} \left(W_{ij}^d + V_{ij}\right)$ serves as a valid *common Lyapunov function* for the underlying hybrid system. Using the analysis of the single integrator case, it is easy to show that at time spaces where no new edges are added, the time derivative of V is given by

$$\dot{V} = -8\left\|\left(\left(P^d \otimes I_2\right) q + (R \otimes I_2)\, q\right)\right\|^2 \leq 0$$

where the P^d matrix is defined as

$$P_{ij}^d = \begin{cases} \sum\limits_{j \neq i} p_{ij}^d, i = j \\ -p_{ij}^d, i \neq j \end{cases}$$

with

$$p_{ij}^d = \begin{cases} \frac{\partial W_{ij}}{\partial \beta_{ij}}, (i,j) \in E \\ \frac{\partial \tilde{W}_{ij}}{\partial \beta_{ij}}, (i,j) \notin E \end{cases}$$

At times when new edges are added, the common Lyapunov function and the control laws of all agents are continuously differentiable while the values of the common Lyapunov function, its time derivative, and the values of the control laws remain *constant*. Hence V serves as a common Lyapunov function for the stability of the Hybrid System and since no Zeno behavior occurs whenever the system enters a new discrete state, i.e. once an edge is added it is never deleted, we can use the extension of LaSalle's invariance principle to hybrid systems established in [25] to show that the system converges to the largest invariant subset of the set $S = \left\{q : \left(\left(P^d + R\right) \otimes I_2\right) q = 0\right\}$. In essence, the results of Theorem 1 and Lemma 2 hold in this case as well. The counterpart of Lemma 3 in the dynamic graph case involves the fact that once an agent j enters the set $N_i^*(t)$ for the first time, it is forced to remain within the sensing zone of i, encoded by the set $N_i(t)$, for all future times. Thus, the definition of edges in the dynamic graph case is meaningful since it respects the sensing capabilities of all agents. The following counterpart of Lemma 3 holds:

Lemma 8: Consider the system of multiple kinematic agents (1) driven by the control law (12). Then, all agent pairs that come into distance less or equal to $d-\varepsilon$ for the first time, remain within distance strictly less than d for all future times.
Proof: Since $V(q(t)) \leq V(q(0)) < \infty$ for all $t \geq 0$ and $V \to \infty$ when $\|q_i - q_j\| \to d$ for at least one pair of agents (i,j) that either (i) have formed an edge at $t=0$, or (ii) have formed an edge at some time τ, $0 \leq \tau \leq t$ we conclude that all pairs of agents that did not initially form an edge and come to a distance less than $d-\varepsilon$ for the first time, remain within distance strictly less than d for all future times. $\diamondsuit$

The fact that agents that initially formed edges remain within distance strictly less than d from each other is established in Lemma 3. These two Lemmas guarantee that the definition of edges in the dynamic graph case respects the limited sensing capabilities of all agents, since each agent has to sense only agents within its sensing zone in order to fulfill the communication link imposed by the existence of edges.

Having now established a framework that allows for addition of edges in the communication graph while maintaining connectivity, we can follow the analysis of the static case to show that similar bounds for the swarm size can be derived in this case as well. In particular, the system now reaches a configuration where equation $\left(\left(P^d + R\right) \otimes I_2\right) q = 0$ holds. Following the analysis of the static graph case, an equation similar to (6) is derived in the dynamic graph case as well:

$$\sum_i \sum_{j \neq i} p_{ij}^d \left\|q_i - q_j\right\|^2 = \sum_i \sum_{j \in M_i} \left|\rho_{ij}\right| \left\|q_i - q_j\right\|^2 \quad (13)$$

An improved result on the bound of the swarm size with respect to the static graph case can be obtained in the dynamic graph case. In particular, using the notation $\left\|q_i - q_j\right\|^2 = \beta_{ij}$, the last equation can be rewritten as

$$\begin{aligned} &\sum_i \left(\sum_{j \in M_i} p_{ij}^d \beta_{ij} + \sum_{j \notin M_i} p_{ij}^d \beta_{ij} \right) = \sum_i \sum_{j \in M_i} \left|\rho_{ij}\right| \beta_{ij} \\ &\Rightarrow \sum_i \sum_{j \notin M_i} p_{ij}^d \beta_{ij} = \sum_i \sum_{j \in M_i} \left(\left|\rho_{ij}\right| - p_{ij}^d\right) \beta_{ij} \end{aligned}$$

Assuming that the repulsion term satisfies the bound $\left|\rho_{ij}\right| \leq \frac{\rho}{\beta_{ij}}$ and noting that $p_{ij}^d \geq 0$ for all $i,j \in \mathcal{N}$, we get

$$\sum_i \sum_{j \in M_i} \left(\left|\rho_{ij}\right| - p_{ij}^d\right) \beta_{ij} \leq \sum_i \sum_{j \in M_i} \left|\rho_{ij}\right| \beta_{ij} \leq \rho \sum_i \left|M_i\right|$$

so that

$$\sum_i \sum_{j \notin M_i} p_{ij}^d \beta_{ij} \leq \rho \sum_i \left|M_i\right| \quad (14)$$

We now denote by $E(\infty)$ the set of edges that have been formed at steady state. Using the bound $p_{ij} \geq a$ on the attractive term for the agents that have formed an edge, the left hand side of the previous inequality is bounded as follows

$$\begin{aligned} &\sum_i \sum_{j \notin M_i} p_{ij}^d \beta_{ij} \\ &= \sum_i \sum_{\substack{j:(i,j) \in E(\infty) \\ j \notin M_i}} p_{ij}^d \beta_{ij} + \sum_i \sum_{\substack{j:(i,j) \notin E(\infty) \\ j \notin M_i}} p_{ij}^d \beta_{ij} \\ &\Rightarrow \sum_i \sum_{j \notin M_i} p_{ij}^d \beta_{ij} \geq \sum_i \sum_{\substack{j:(i,j) \in E(\infty) \\ j \notin M_i}} p_{ij}^d \beta_{ij} \\ &= \sum_i \sum_{\substack{j:(i,j) \in E(\infty) \\ j \notin M_i}} p_{ij} \beta_{ij} \end{aligned}$$

$$\Rightarrow \sum_i \sum_{j \notin M_i} p_{ij}^d \beta_{ij} \geq \sum_i \sum_{\substack{j:(i,j) \in E(\infty) \\ j \notin M_i}} a d_1^2 \quad (15)$$

The two bounds (14),(15) suggest that

$$\sum_i \sum_{\substack{j:(i,j) \in E(\infty) \\ j \notin M_i}} a d_1^2 \leq \rho \sum_i \left|M_i\right| \quad (16)$$

at steady state. We will now show that an appropriate choice of d_1 forces all agents that have formed an edge to be at a distance not larger than d_1 at steady state, i.e. $\|q_i - q_j\| \leq d_1$, for all $(i,j) \in E(\infty)$. This is proved by showing that the inequality (16) is not viable even in the worst case scenario. Thus, let us assume that at least one pair that has formed an edge at steady state is at a distance larger than d_1 from one another, i.e. $\|q_k - q_l\| > d_1$ for some $(k,l) \in E(\infty)$. This implies that $k \notin M_l$ and vice versa. In that case (16) yields

$$\begin{aligned} &\sum_i \sum_{\substack{j:(i,j) \in E(\infty) \\ j \notin M_i}} a d_1^2 = 2 a d_1^2 \leq \sum_i \sum_{j \notin M_i} p_{ij}^d \beta_{ij} \leq \rho \sum_i \left|M_i\right| \\ &= \rho \left\{ \left(N-1\right)\left(N-2\right) + 2\left(N-2\right) \right\} = \rho \left(N^2 - N - 2\right) \\ &\Rightarrow d_1^2 \leq \frac{\rho\left(N^2 - N - 2\right)}{2a} \end{aligned}$$

The last inequality is rendered impossible by choosing $d_1^2 > \frac{\rho\left(N^2 - N - 2\right)}{2a}$. In this case, we have $j \in M_i$ for all pairs of agents that form an edge at steady state, and hence an ultimate bound is given by

$$\beta_{ij} \leq d_1^2, \forall \left(i,j\right) \in E\left(\infty\right) \quad (17)$$

This equation provides the means to provide a better bound of the swarm size, as will be shown in the sequel. We first note that the parameter d_1 can be chosen by the following inequality:

$$\frac{\rho\left(N^2 - N - 2\right)}{2a} < d_1^2 < \frac{\rho N\left(N-1\right)}{a} \quad (18)$$

The last inequality is feasible since the inequality $\frac{\rho\left(N^2 - N - 2\right)}{2a} < \frac{\rho N (N-1)}{a}$ is equivalent to $N^2 - N + 2 > 0$ which holds for all $N > 0$.

The following theorem, which is the counterpart of Theorem 4 in the dynamic graph case, shows that a better bound is derived in the dynamic graph case provided that d_1 satisfies (18):

Theorem 9: Assume that the swarm (1) evolves under the control law (12) and the initially formed communication graph is connected. Denote by $\beta_{\max}^d$ the maximum distance between two members of the group, i.e. $\beta_{\max}^d = \max_{i,j\in\mathcal{N}} \|q_i - q_j\|^2$. Assume that the parameter d_1 satisfies (18). Under the preceding assumptions, the following bound holds at steady state: $\beta_{\max}^d \leq d_1^2 (N-1)$. We moreover have $\beta_{\max}^d < \beta_{\max}$ where $\beta_{\max} = \frac{\rho}{a} N (N-1)^2$ is the swarm size corresponding to the static graph case of Theorem 4.

Proof: Since the graph is connected, the maximum length of a path connecting two arbitrary vertices is $N-1$. Equation (17) now yields $\beta_{\max}^d \leq d_1^2 (N-1)$. Now since d_1 satisfies (18), we have $\beta_{\max}^d \leq d_1^2 (N-1) < \frac{\rho N (N-1)^2}{a} = \beta_{\max}$, and thus, $\beta_{\max}^d < \beta_{\max}$. $\diamondsuit$

This result shows that allowing edges to be added in a dynamic fashion, leads to an improved (i.e. smaller) swarm size. This derivation is not surprising, since the addition of new communication links increases the attractive potential and hence leads to a tighter swarm size.

VII. Simulations

To support the results of this work we provide a series of computer simulations.

The first simulation of Figure 2 involves the evolution of a swarm of nine single integrator agents that navigate under the proposed control law in both the static and dynamic edge addition cases. In both cases the agents have the same initial conditions and controller parameters. In particular, agents use an unbounded repulsive potential. The first screenshot shows the initial positions of the nine agents. In the first case in the middle they navigate under the control law (4) while in the second case at the bottom under the control law (12). The parameters in the simulations are given by $d_1 = 0.033$, $d = 0.04$, $\frac{\rho}{a} = 2 \cdot 10^{-5}$ and of course, $N = 9$. This choice of d renders the initially formed communication graph connected. Moreover, condition (18) is satisfied since $\frac{\rho(N^2-N-1)}{2a} = 7 \cdot 10^{-4}$, $\frac{\rho N(N-1)}{a} = 14.4 \cdot 10^{-4}$, and $d_1^2 = 10.89 \cdot 10^{-4}$. Thus $\frac{\rho\left(N^2-N-2\right)}{2a} < d_1^2 < \frac{\rho N(N-1)}{a}$ holds. This is a sufficient condition for the fact that the swarm size is smaller in the case of dynamic edge addition.

As witnessed in Figure 2 the control law in the dynamic graph formulation indeed leads to a tighter final swarm size. In fact, in the dynamic case the edges are added until the graph is rendered complete, i.e., we have all-to-all communication, while in the static graph case the initially formed graph remains invariant for all time.

A comparison of the final swarm sizes of the two cases is depicted in Figure 3. This figure shows the evolution of the swarm size in both cases from time 1000 and onwards. Note that the swarm size in the static graph case is bounded by $\beta_{\max} \leq \frac{\rho}{a} N(N-1)^2 = 0.01152$ while in the dynamic case by $\beta_{\max}^d \leq d_1^2(N-1) = 0.008089$ by virtue of Theorems 4 and 9 respectively. It can be verified from the two plots of Figure 3 that the final swarm size in both cases fulfills the expected bounds. It can also be witnessed that apart from the reduction of the swarm size, the convergence rate is also significantly increased in the case of the dynamic edge addition. This is depicted by the significantly smaller swarm size the team has attained at time 1000 in the second case.

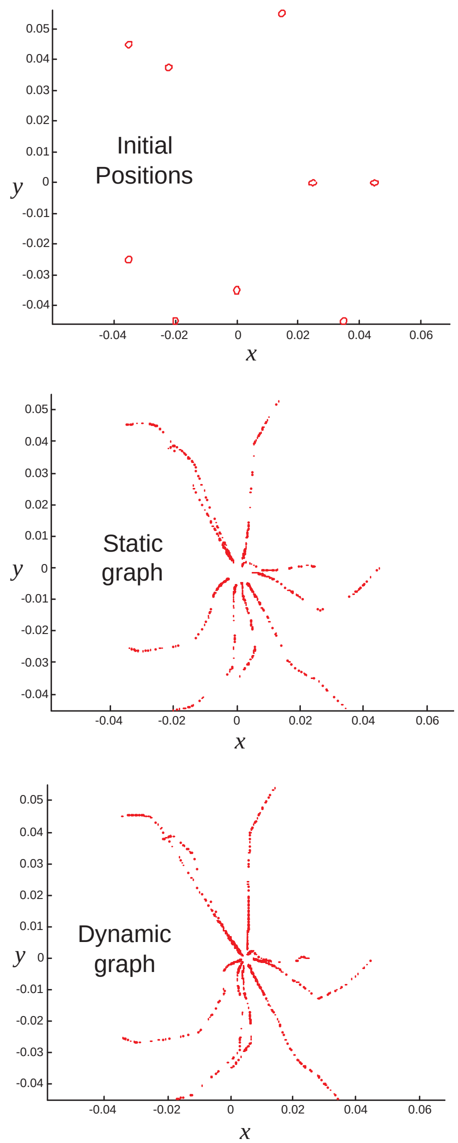

Fig. 2. Evolution in time of the swarm under the control law (4), for the static communication graph case, in the middle, and the control law (12), for the dynamic graph case, at the bottom. The communication graph is connected in both cases. The second control law leads to a smaller swarm size.

The same values of controller parameters have been retained in the simulation of Figure 4 as well. We have only decreased the sensing radius with respect to the first simulation. In particular, we now have $d = 0.035$. Agents navigate under the static graph control strategy (4). The initial positions of the agents of this simulation are the same as in the previous one while Figure 4 depicts the evolution of the closed-loop system in time. This decrease renders the initially formed communication graph disconnected. Specifically, there are two connected components. Due to the lack of connectivity, the

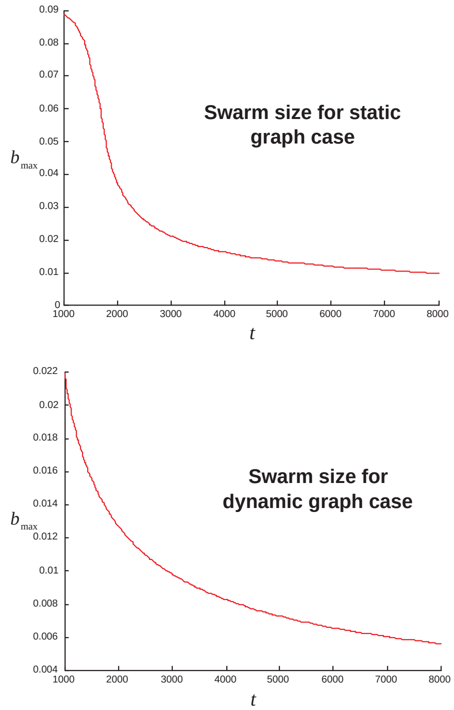

Fig. 3. Evolution of the swarm size for the two simulations of Figure 2. The dynamic graph formulation leads to a smaller swarm size.

swarm is eventually split into its two connected components, as witnessed in the figure.

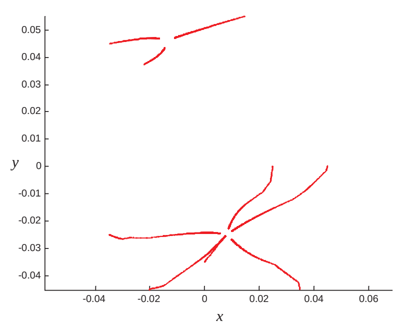

Fig. 4. Lack of connectivity in the initially formed communication graph decouples the swarm into its connected components.

The last simulation in Figure 5 involves evolution of a swarm of six kinematic unicycles navigating under the control law laws (9),(10). The first screenshot shows the initial positions of the six agents while the second one the evolution of their trajectories in time. Swarm aggregation is eventually achieved, since the communication graph that is formed based on the *initial* relative positions of the agents, is connected. The same values of controller parameters as in the first simulation have been retained in the simulation of Figure 5 as well.

VIII. Conclusions

A distributed control strategy for connectivity preserving swarm aggregation with collision avoidance was presented.

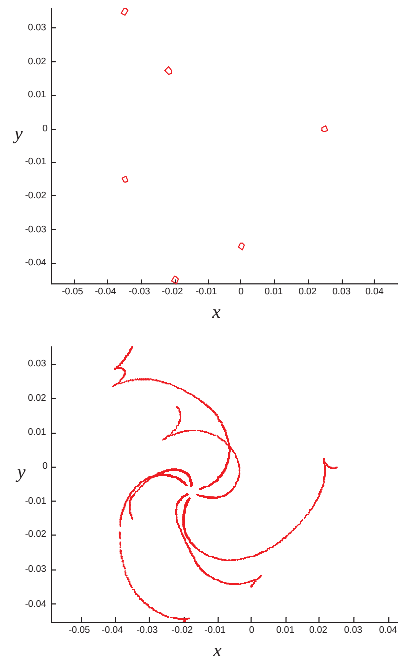

Fig. 5. Evolution in time of the nonholonomic swarm under the control laws (9),(10). The communication graph is connected.

Specifically, each agent was assigned with a control law which was the sum of two elements: a repulsive potential field, which was responsible for the collision avoidance objective, and an attractive potential field, that forced the agents to converge to a configuration where they are close to each other. Furthermore, the attractive potential field forced the agents that were initially located within the sensing radius of an agent to remain within this area for all time. It was shown that under the proposed control law agents converge to a configuration where each agent is located at a bounded distance from each of its neighbors. In the case of dynamic edge addition, an improved bound on the swarm size was derived. The results were extended to deal with the case of nonholonomic kinematic unicycle-type agents as well.

Further research involves the development of bounded control laws for connectivity maintenance, as opposed to the unbounded control laws used in this paper. The use of bounded control laws can be more practical in some problems where actuation is required to be bounded. Furthermore, we aim to extend the results to dynamic agents and take the individual robots size into account in the collision avoidance procedure.

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