Self-Triggered Sampling for Second-Moment Stability of State-Feedback Controlled SDE Systems

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Abstract

Event-triggered and self-triggered control, whereby the times for controller updates are computed from sampled data, have recently been shown to reduce the computational load or increase task periods for real-time embedded control systems. In this work, we propose a self-triggered scheme for nonlinear controlled stochastic differential equations with additive noise terms. We find that the family of trajectories generated by these processes demands a departure from the standard deterministic approach to event- and self-triggering, and, for that reason, we use the statistics of the sampled-data system to derive a self-triggering update condition that guarantees second-moment stability. We show that the length of the times between controller updates as computed from the proposed scheme is strictly positive and provide related examples.

Key words: stochastic control; stochastic systems; feedback stabilization; real-time systems; embedded systems

1 Introduction

The implementation of nonlinear feedback controllers on digital computer platforms necessitates a choice of the times at which the controller should be updated. In traditional setups, these updates are scheduled periodically with time periods that under all circumstances do not compromise closed-loop stability. However, a longer period between controller updates offers greater design flexibility and increases the availability of computational resources. Having that in mind, a more suitable approach for scheduling controller updates would take into account system states and perform updates only when necessary. Along these lines, the approaches of event- and self-triggering have recently been proposed to lengthen the intervals between updates without sacrificing stability [1, 5, 7, 8, 13, 16, 17, 19, 22–24]. The time at which the control is updated hinges upon a state-dependent criterion in a way that ensures stability of the closed-loop feedback control system under the outdated samples. The intervals between updates vary and may be longer than those under a periodic implementation [7].

In an event-triggered implementation, the system state is updated when it deviates from the previous sample by a sufficient amount [8, 17, 19, 22]. This requires continuous monitoring of the system state, which may be impractical. Therefore, in this paper, we consider a self-triggered approach, in which the decision to update is computed from *predictions* of when the system state will deviate by a given threshold from the last update without compromising the closed-loop stability [5, 13, 16, 23]. However, for systems under the influence of disturbances or noise, it may be more difficult to make these predictions or to ensure the intended stability results. Along these lines, the robustness of a self-triggered control strategy to disturbances was analyzed in [16] for linear systems. In [23], a self-triggered $\mathcal{H}_\infty$ control under state-dependent disturbances was developed for linear systems, and this was extended in [24] for an exogenous disturbance in $\mathcal{L}_2$ space. During the review of this paper, we were also made aware of very recent event- and self-triggered implementations of linear quadratic controllers [6, 9]. All of the above works provide event-and self-triggered control solutions in the realm of linear control, but nonlinear stochastic systems remain largely untouched.

In this work, we develop a self-triggered scheme for the control of nonlinear stochastic dynamical systems de-

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scribed by stochastic differential equations (SDEs) [15] with additive noise terms [2]. In the area of control, this type of dynamical model is commonly used for Kalman-, or Extended Kalman-filter designs [10]. For systems of this type, not only can it be difficult to predict the system state at a future time, but the noise may drastically alter the system dynamics. Because of that, we develop an update rule based on certain statistics of the state distribution that guarantees p-moment stability [11, 15] ($p > 0$) of the SDE solution. (While our results are valid for general $p > 0$, the second moment [$p = 2$] is most convenient for our numerical examples, and we focus primarily on second-moment stability.) The length of time between controller updates is shown to be strictly positive and can be computed from the previously-measured state. Since the intensity of the noise in many stochastic systems is independent of the state, we split our analysis into two parts based on whether or not the noise diminishes as the system state approaches a stable equilibrium.

We recently became aware of another attempt at self-triggered stabilization of nonlinear stochastic control systems. In [2], the authors develop a self-triggering rule for SDEs that guarantees stability in probability (which is weaker than p-moment stability). However, no examples with nonlinear stochastic event- or self-triggered task durations have so far been presented in literature. (The example in [2] is reformulated as a safety problem in order to satisfy that paper's assumptions.) Comparisons of task durations in the deterministic event- and self-triggered literature have proven revealing as new works arise, and so we provide two stochastic examples with task update periods for future work comparisons. The first example is a stochastic version of a deterministic problem [22], and the second is a robotics-relevant stochastic wheeled cart control problem whose motivation is described in [3], for instance. These practically-relevant examples are another contribution of this paper.

This paper is organized as follows: In Section 2, we formulate the problem and call attention to why the standard deterministic approach for event- and self-triggering may not apply to stochastic systems. Section 3 proposes a self-triggering scheme with strictly positive times between control updates, which is illustrated in Section 4 with numerical examples. Finally, Section 5 summarizes the results of this paper and provides directions for future research.

[2] A preliminary version of this work with a more complicated triggering scheme originally appeared in [4].

2 Problem Statement and Preliminaries

We will consider state-feedback control systems defined by stochastic differential equations [18] of the form

$$\mathrm{d}x(t) = f(x,u)\mathrm{d}t + g(x,u)\mathrm{d}w, \qquad x \in \mathbb{R}^n, \quad (1)$$

where $\mathrm{d}w$ is a (multi-dimensional) increment of a standard Wiener process, $u(t) : [0,\infty) \to \mathbb{R}^m$ is a control input, and $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are the drift and diffusion scaling factors of the dynamics. Usually these systems are formally defined alongside a complete probability space $(\Omega, \mathcal{F}, P)$ [18], where Ω is the set of possible outcomes, $\mathcal{F}$ is a filtration, and $P : \mathcal{F} \to [0,1]$ is a probability measure function. We consider the system (1) with sample-and-hold state measurements, i.e.,

$$\mathrm{d}x(t) = f(x,u)\mathrm{d}t + g(x,u)\mathrm{d}w \quad (2)$$
$$u(t) = k(x_i), \qquad t \in [t_i, t_{i+1}) \quad (3)$$

where t_i, $i = 0, 1, \ldots$, is a sequence of update, or triggering, times, and $x_i = x(t_i)$ is the corresponding sequence of measurements used to update the feedback control $k(x_i)$. The error signal $e(t)$ is defined as

$$e(t) = x_i - x(t), \qquad t \in [t_i, t_{i+1})\,. \quad (4)$$

Then (2) is

$$\mathrm{d}x(t) = f(x, k(x+e))\mathrm{d}t + g(x, k(x+e))\mathrm{d}w \quad (5)$$

To simplify notation in the sequel, we will write with a slight abuse of notation $f(x,e)$ instead of $f(x,k(x+e))$ and $g(x,e)$ instead of $g(x,k(x+e))$.

Our goal is to develop an update rule for the stochastic system (4)-(5) based on the observable state x_i, that will render the system stable (in some sense to be described shortly) and guarantee strictly positive time between sampling time points, that is, $t_{i+1} - t_i > 0$, $i = 0, 1, \ldots$. Typically this is performed by examining the time for which the error $e(t)$ remains below some threshold; see, for example, [22]. However, in the stochastic case considered in this work, the error may exceed this bound instantaneously, that is, for any $M < \infty$ and any time $t > 0$, the Euclidean norm $|e| = \sqrt{e^\intercal e}$ of a solution $e(t)$ to a stochastic differential equation will exceed the level M with non-zero probability, or $\Pr(|e(t)| \geq M) > 0$ [18, Exercise 8.13]. In other words, although certain trajectories of $e(t)$ for fixed $\omega \in \Omega' \subset \Omega$ may remain below a given threshold for a sufficiently large time, the same can not be said about all trajectories $e(t)$ (or $x(t)$) defined by (4)-(5). Additionally, the second-order differential terms required in stochastic evolution equations can cause certain quantities found in deterministic literature (e.g., the quantity $|e(t)|/|x(t)|$ in [22]) to experience unbounded growth near the origin. These facts make it difficult to develop a sampling rule using the trajectories

$e(t)$ and $x(t)$ or predictions of these processes. Because of this, we instead consider the p-th moments of these processes, $\mathbb{E}(|e|^p)$ and $\mathbb{E}(|x|^p)$, with respect to the initial state $x(t_i)$ and error $e(t_i) = 0$. Based on these statistics, we develop a triggering condition to guarantee the stability of $x(t)$ in the p-th moment [15]:

Definition 2.1 (cf. [14]) *A system is said to be* practically p-moment stable *if there exist a class* $\mathcal{K}$ *function*[3] γ, *a class* $\mathcal{KL}$ *function* β, *and a constant* $d \geq 0$ *such that for all* $t \geq 0$,

$$\mathbb{E}\left(|x(t)|^p\right) \leq \beta\left(\mathbb{E}\left(|x(0)|^p\right), t\right) + \gamma(d).$$

If $d = 0$, *the system is said to be* p-moment stable *(see Definition 2.1 in [11]).*

Definition 2.2 (cf. [14]) *A system is said to be* p-moment input-to-state stable (ISS) *with respect to an error* $e(t)$ *if there exist a class* $\mathcal{KL}$ *function* β *and class* $\mathcal{K}$ *functions* γ *and* λ *such that for all* $t \geq 0$,

$$\mathbb{E}\left(|x(t)|^p\right) \leq \beta\left(\mathbb{E}\left(|x(0)|^p\right), t\right) + \gamma\left(\mathbb{E}\left(\lambda(\sup_{t\geq 0}|e(t)|)\right)\right). \tag{6}$$

We begin with a characterization of p-moment input-to-state stability [21] of the closed-loop feedback control system with respect to errors caused by outdated samples based on the following theorem.

Theorem 1 *Suppose there exist a convex class* $\mathcal{K}_\infty$ *function* $\underline{\alpha}$, *a class* $\mathcal{K}_\infty$ *function* $\overline{\alpha}$, *a non-negative function* α, *and non-negative function* $V(x,t)$ *that is twice differentiable in its first argument such that*

$$\underline{\alpha}(|x|^p) \leq V(x,t) \leq \overline{\alpha}(|x|^p), \tag{7}$$
$$\mathbb{E}\mathcal{L}V(x,t) \leq \mathbb{E}\left(\lambda(|e|)\right) - \mathbb{E}\left(\alpha(x)\right) \tag{8}$$

for all $t \geq 0$, *where where the differential operator* $\mathcal{L}$, *when applied to* $V(x,t)$, *is*

$$\mathcal{L}V(x,t) = \frac{\partial V}{\partial t} + f^\intercal \frac{\partial V}{\partial x} + \frac{1}{2}\mathit{Trace}\left(g^\intercal \frac{\partial^2 V}{\partial x^2} g\right), \tag{9}$$

and where $\lim_{|x|\to\infty} \alpha(x)/\overline{\alpha}(|x|^p) > 0$. *Then the system (2)-(4) is p-moment ISS with respect to the error* e.

[3] A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times [0,\infty) \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if, for each fixed t, $\beta(x,t)$ is of class $\mathcal{K}$, and, for each fixed x, $\beta(x,t)$ is decreasing with $\beta(x,t) \to 0$ as $t \to \infty$.

Theorem 1 is a specific case of [11, Theorem 3.1]. The latter applies to more general systems with delays and Markovian switching. That the Lyapunov characterization we choose to use arises from the study of stochastic differential *delay* equations should not come as a surprise, since a sample x_i really amounts to a delayed state variable.

If the error were to satisfy (cf. [22])

$$\mathbb{E}\left(\lambda(|e|)\right) \leq \theta\mathbb{E}\left(\alpha(|x|)\right), \qquad 0 < \theta < 1, \tag{10}$$

then by (8), the state $x(t)$ is p-moment stable since

$$\mathbb{E}\mathcal{L}V \leq -(1-\theta)\mathbb{E}\alpha(|x|). \tag{11}$$

In previous works in deterministic literature, the duration for which the error $e(t)$ satisfies a condition like (10) [22, 23] is found (but without expectation operators), either using the value of the state (for an event-triggered scheme) or using predictions (a self-triggered scheme). However, as a consequence of using the moments for the update rule, we must rule out an event-triggered implementation in our approach. From a practical standpoint, the controller can only measure an individual sample path of the process $x(t)$, and not the statistics $\mathbb{E}(\lambda(|e|))$ or $\mathbb{E}(\alpha(|x|))$. However, the latter quantities can be predicted on the interval $[t_i, t_{i+1})$ based on the last-sampled state x_i, and are therefore suitable for a self-triggered approach.

In many systems, the diffusion scaling $g(\cdot,\cdot)$ is independent of the state and control and does not vanish at the origin, i.e., $g(0,0) \neq 0$. For systems of this type, the second-order term in (9) may be constant. In this work, we treat that constant as an additional disturbance that acts alongside the error due to sampling $e(t)$. We therefore consider two stability results. If $g(0,0) = 0$, that is, if the noise vanishes at the origin, then we will seek p-moment stability. Otherwise, we consider practical p-moment stability.

3 Triggering Condition

Suppose that there is a Lyapunov function for the system (2)-(4) satisfying (7) and

$$\mathbb{E}\mathcal{L}V(x,t) \leq \mathbb{E}\left(\lambda(|e|)\right) - \mathbb{E}\left(\alpha(x(t))\right) + d \tag{12}$$

Here, $d > 0$ allows for the possibility of a constant disturbance that does not diminish at the origin. This may occur, for example, if the second-order term in (9) includes an additive constant. In our approach, we use this secondary disturbance to our advantage and include it in the update rule. Next, suppose that the error were to satisfy (cf. (10))

$$\mathbb{E}(\lambda(|e|)) \leq \theta\mathbb{E}(\alpha(x)) + \theta_d d, \tag{13}$$

for a constant $0 < \theta < 1$ and parameter $\theta_d > 0$. Then from (12),

$$\mathbb{E}\mathcal{L}V \leq -(1-\theta)\mathbb{E}(\alpha(x)) + (1+\theta_d)d. \tag{14}$$

In this case, (14) does not include the error $e(t)$ due to the sampling rule, but does include the constant disturbance $(1+\theta_d)d$. By rewriting (8) with $\lambda(|e|)$ replaced by $(1+\theta_d)d$ and using Theorem 1, one can show that the system will be practically p-moment stable with $\gamma(d)$ in Eq. (6) replaced by $\gamma((1+\theta_d)d)$. If $(1+\theta_d)d = 0$, i.e., if $d = 0$, then the system will be p-moment stable. We now turn to the task of determining how long the condition (13) holds true based on the last-observed state x_i.

3.1 Predictions of the moments of the processes

The following lemma relates the upper and lower bounds of the statistics $\mathbb{E}(|e(t)|^2)$ and $\mathbb{E}(|x(t)|^2)$, respectively, which are not observable, to the norm of the last-observed system state x_i. Note that while these bounds are valid for the $p = 2$ moment, they will later be used in our self-triggering rule to develop stability in the p-th moment, that is, for general $p > 0$, according to Definition 2.1. We assume monotone growth and local Lipschitz continuity of the SDE (1), and we stress that these assumptions are no stronger than those used commonly to prove existence and uniqueness of the SDE solution $x(t)$ (cf. [15, Theorem 2.3.5] with $e = 0$):

Assumption 1. There exists a positive constant K such that for all $x, e \in \mathbb{R}^n$,

$$x^\intercal f(x,e) + \frac{1}{2}|g(x,e)|^2 \leq K(1 + |x|^2 + |e|^2) \tag{15}$$

Assumption 2. Using the notation $a \vee b = \max\{a,b\}$, for every integer $m \geq 1$, there exists a positive constant L_m such that for all $x, e, x', e' \in \mathbb{R}^n$ with $|x| \vee |x'| \vee |e| \vee |e'| \leq m$,

$$\begin{aligned}|f(x,e) - f(x',e')|^2 &\vee |g(x,e) - g(x',e')|^2 \\ &\leq L_m(|x-x'|^2 + |e-e'|^2)\end{aligned} \tag{16}$$

Lemma 2 *Assume the monotone growth condition (15) and Lipschitz continuity (16). If the system state has been updated as $x_i = x(t_i)$, then for any $t \in [t_i, t_{i+1})$, the means of the norms $|e(t)|^2$ and $|x(t)|^2$ are upper and lower bounded, respectively, based on the following inequalities*

$$\mathbb{E}(|e(t)|^2) \leq A(|x_i|, t-t_i) \tag{17}$$
$$\mathbb{E}(|x(t)|^2) \geq B(|x_i|, t-t_i) \tag{18}$$

where

$$A(|x_i|, t-t_i) = \frac{2|x_i|^2+1}{3}\left(e^{12K(t-t_i)} - 1\right) \tag{19}$$
$$B(|x_i|, t-t_i) = \frac{5|x_i|^2+1}{3}e^{-6K(t-t_i)} - \frac{2|x_i|^2+1}{3}. \tag{20}$$

If $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ admit an equilibrium point, i.e., $f(0,0) = g(0,0) = 0$, then we additionally have that

$$\mathbb{E}(|e(t)|^2) \leq C(|x_i|, t-t_i) \tag{21}$$
$$\mathbb{E}(|x(t)|^2) \geq D(|x_i|, t-t_i) \tag{22}$$

where

$$\begin{aligned}&C(|x_i|, t-t_i) \\ &\quad = \frac{2|x_i|^2(1+\sqrt{L_m})}{4+3\sqrt{L_m}}\left(e^{(4\sqrt{L_m}+3L_m)(t-t_i)} - 1\right)\end{aligned} \tag{23}$$

$$\begin{aligned}&D(|x_i|, t-t_i) \\ &\quad = \frac{|x_i|^2}{4+3\sqrt{L_m}}\Big((6+5\sqrt{L_m})e^{-(4\sqrt{L_m}+3L_m)(t-t_i)} \\ &\qquad -2(1+\sqrt{L_m})\Big).\end{aligned} \tag{24}$$

PROOF. The total differential of $\mathbb{E}(|e|^2)$ can be found using Itô's Lemma and (4) as

$$\begin{aligned}\mathrm{d}\mathbb{E}(|e|^2) &\leq \mathrm{d}\mathbb{E}\left(2|x_i|^2 + 2|x|^2\right) \\ &= 2\mathbb{E}\left(2x^\intercal f(x,e)\mathrm{d}t + 2x^\intercal g(x,e)\mathrm{d}w + |g(x,e)|^2\mathrm{d}t\right) \\ &= 4\mathbb{E}\left(x^\intercal f(x,e) + \frac{1}{2}|g(x,e)|^2\right)\mathrm{d}t\end{aligned}$$

From the monotone growth condition (15)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(|e|^2) &\leq 4K\left(1 + \mathbb{E}(|x|^2) + \mathbb{E}(|e|^2)\right) \\ &\leq 4K\left(1 + \mathbb{E}(|x_i - e|^2) + \mathbb{E}(|e|^2)\right) \\ &\leq 4K\left(1 + 3\mathbb{E}(|e|^2) + 2\mathbb{E}(|x_i|^2)\right) \\ &\leq 12K\mathbb{E}(|e|^2) + 8K|x_i|^2 + 4K\end{aligned}$$

Applying the comparison principle in [12], along with the fact that $e(t_i) = 0$, we obtain (17). For the second inequality (18), we can obtain in a similar manner

$$\begin{aligned}\left|\mathrm{d}\mathbb{E}(|x|^2)\right| &\leq \big|\mathbb{E}\big(2x^\intercal f(x,e)\mathrm{d}t + 2x^\intercal g(x,e)\mathrm{d}w \\ &\qquad + |g(x,e)|^2\mathrm{d}t\big)\big| \\ &= 2\left|\mathbb{E}\left(x^\intercal f(x,e) + \frac{1}{2}|g(x,e)|^2\right)\mathrm{d}t\right| \\ &\leq 6K\mathbb{E}(|x|^2)\mathrm{d}t + 4K|x_i|^2\mathrm{d}t + 2K\mathrm{d}t,\end{aligned}$$

so that $\frac{\mathrm{d}}{\mathrm{d}t}\left(-\mathbb{E}(|x|^2)\right) \leq -6K\left(-\mathbb{E}(|x|^2)\right)+4K|x_i|^2+2K$. Using the comparison principle with $x(t_i) = x_i$ yields (18).

Next, recall that from (2) and (4), the error kinematics satisfies for $t \in [t_i, t_{i+1})$:

$$e(t) = -\int_{t_i}^{t} f(x,u)\mathrm{d}s - \int_{t_i}^{t} g(x,u)\mathrm{d}w_s. \tag{25}$$

Using Itô's Lemma and the inequality $2ab \leq a^2 + b^2$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(|e|^2) = -2\mathbb{E}\left(e^{\intercal}f(x,e)\right) + \mathbb{E}(|g(x,e)|^2)\\
&\leq 2\mathbb{E}\left(\left[L_m^{\frac{1}{4}}|e|\right]\left[\frac{|f(x,e)|}{L_m^{\frac{1}{4}}}\right]\right) + \mathbb{E}(|g(x,e)|^2)\\
&\leq \sqrt{L_m}\mathbb{E}\left(|e|^2\right) + \frac{1}{\sqrt{L_m}}\mathbb{E}\left(|f(x,e)|^2\right) + \mathbb{E}(|g(x,e)|^2)
\end{aligned}$$

Inserting the Lipschitz condition (16) with $x' = e' = 0$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(|e|^2)\\
&\quad\leq \sqrt{L_m}\mathbb{E}\left(|e|^2\right) + (\sqrt{L_m} + L_m)\mathbb{E}\left(|x_i - e|^2 + |e|^2\right)\\
&\quad\leq (4\sqrt{L_m} + 3L)\mathbb{E}(|e|^2) + 2(\sqrt{L_m} + L_m)|x_i|^2.
\end{aligned}$$

From the comparison principle and the fact that $e(t_i) = 0$, we obtain (21). The final inequality (22) follows similarly. □

3.2 A Self-triggering Sampling Rule

With the inequalities from Lemma 2 on $\mathbb{E}(|e(t)|^2)$ and $\mathbb{E}(|x(t)|^2)$, we are ready to state our main results. The following theorem provides relations based on (13) that can be used to calculate a strictly positive inter-execution time $\tau_i = t_{i+1} - t_i > 0$ as a function of the last-observed state x_i.

Theorem 3 *Assume that in addition to the conditions of Theorem 1 and (15)-(16), there exist a convex class $\mathcal{K}$ function $\alpha_v(\cdot)$ and a concave class $\mathcal{K}$ function $\lambda_c(\cdot)$ which satisfy*

$$\alpha_v(2|x|^2) \leq 2\alpha(x) \tag{26}$$

$$\lambda_c(|e|) \geq \lambda(\sqrt{|e|}). \tag{27}$$

Suppose that the system (2)-(4) has been updated at $t = t_i$ with state x_i, and that the time until the next update $\tau_i = t_{i+1} - t_i$ is such that

$$\lambda_c\left(A(|x_i|, \tau_i)\right) \leq \theta\alpha_v\left(B(|x_i|, \tau_i) + d_\alpha\right). \tag{28}$$

Then (13) will hold with

$$\theta_d d = \theta\alpha_v(2d_\alpha)/2, \tag{29}$$

ensuring practical p-moment stability. Moreover, if $d_\alpha > 0$ and $|x_i| \leq \bar{x} < \infty$, the execution times do not become arbitrarily close and do not reach an accumulation point, i.e., there exists a time $\tau > 0$ such that $\tau_i \geq \tau$, $i = 0, 1, \ldots$.

In the case that $f(0,0) = g(0,0) = 0$, and if the time until the next update τ_i is such that

$$\lambda_c\left(C(|x_i|, \tau_i)\right) \leq \theta\alpha_v\left(D(|x_i|, \tau_i) + d_\alpha\right) \tag{30}$$

then (13) will hold with $\theta_d d = \theta\alpha_v(2d_\alpha)/2$ (and $d = 0$ if $d_\alpha = 0$), ensuring practical p-moment stability if $d_\alpha > 0$ or p-moment stability if $d_\alpha = 0$. Moreover, for any non-negative d_α and $|x_i| \leq \bar{x} < \infty$, the execution times do not reach an accumulation point, i.e., there exists a time $\tau > 0$ such that $\tau_i \geq \tau$, $i = 0, 1, \ldots$.

PROOF. Substitution of the inequalities (17) and (18) into (28) gives

$$\lambda_c\left(\mathbb{E}(|e|^2)\right) \leq \theta\alpha_v\left(\mathbb{E}\left(|x|^2\right) + d_\alpha\right). \tag{31}$$

Since $\alpha_v(\cdot)$ is convex, we have from (26) and Jensen's inequality that the right hand side of (31) is

$$\begin{aligned}
&\theta\alpha_v\left(\mathbb{E}\left(|x|^2\right) + d_\alpha\right)\\
&\qquad= \theta\alpha_v\left(2\left(\frac{1}{2}\mathbb{E}(|x|^2) + \frac{1}{2}d_\alpha\right)\right)\\
&\qquad\leq \theta\mathbb{E}\left(\alpha_v(2|x|^2)/2\right) + \theta\alpha_v(2d_\alpha)/2\\
&\qquad\leq \theta\mathbb{E}\left(\alpha(|x|)\right) + \theta_d d
\end{aligned} \tag{32}$$

which is the right hand side of (13) with $\theta_d d = \theta\alpha_v(2d_\alpha)/2$. Similarly, through the concavity of $\lambda_c(\cdot)$ and (27), the left hand side of (31) can also be made to match that of (13) using the assumption $\lambda_c\left(\mathbb{E}(|e|^2)\right) \geq \mathbb{E}\left(\lambda(|e|)\right)$:

$$\mathbb{E}\left(\lambda(|e|)\right) \leq \mathbb{E}\left(\lambda_c(|e|^2)\right) \leq \lambda_c\left(\mathbb{E}(|e|^2)\right).$$

In summary, we have shown that (28) implies (13), i.e.,

$$\begin{aligned}
\mathbb{E}\left(\lambda(|e|)\right) &\leq \lambda_c\left(A(|x_i|, \tau_i)\right)\\
&\leq \theta\alpha_v\left(B(|x_i|, \tau_i) + d_\alpha\right) \leq \theta\mathbb{E}\left(\alpha(|x|)\right) + \theta_d d.
\end{aligned}$$

Next, to show the existence of a lower bound $\tau_i \geq \tau > 0$, $i = 0, 1, \ldots$, for the inter-execution times implicitly defined by (28), let us define a function $\kappa(|x_i|)$ that is chosen to satisfy

$$0 < \kappa(|x_i|) < |x_i|^2 + d_\alpha. \qquad \text{for all } |x_i| \geq 0 \tag{33}$$

Clearly such a $\kappa(\cdot)$ exists, and, moreover, $\kappa(|x_i|) > A(|x_i|, 0)$ and $\kappa(|x_i|) < B(|x_i|, 0) + d_\alpha$. We need to verify that there exists a τ_i such that

$$\begin{aligned}\lambda_c\left(A(|x_i|,0)\right) &< \lambda_c\left(\kappa(|x_i|)\right)\\ &\le \lambda_c\left(A(|x_i|,\tau_i)\right) \le \theta\alpha_v\left(B(|x_i|,\tau_i)+d_\alpha\right)\\ &\le \alpha_v\left(\kappa(|x_i|)\right) < \alpha_v\left(B(|x_i|,0)+d_\alpha\right).\end{aligned}$$

By solving for τ_i in both $\kappa(|x_i|) \le A(|x_i|, \tau_i)$ and $\kappa(|x_i|) \ge B(|x_i|, \tau_i) + d_\alpha$, we obtain

$$\begin{aligned}\tau_i \ge \frac{1}{12K}\ln\left(1+\kappa(|x_i|)\frac{3}{2|x_i|^2+1}\right)\\ \wedge\ -\frac{1}{6K}\ln\left(1-\frac{3(|x_i|^2+d_\alpha-\kappa(|x_i|))}{5|x_i|^2+1}\right) > 0 \quad (34)\end{aligned}$$

where we use the notation $a \wedge b = \min\{a, b\}$, and where the strict positivity of the right hand side is due to (33).

In the case of an equilibrium point $f(0, 0) = g(0, 0) = 0$, the desired relation (13) follows in the same way as in the first part of this proof with the use of $C(|x_i|, \tau_i)$ and $D(|x_i|, \tau_i)$ in place of $A(|x_i|, \tau_i)$ and $B(|x_i|, \tau_i)$. For the lower bound on τ_i, choose a function $\kappa(|x_i|)$ to satisfy

$$0 < \kappa(|x_i|) < |x_i|^2 + d_\alpha, \qquad \text{for all } |x_i| > 0 \qquad (35)$$

$$\lim_{|x_i|\to 0^+}\frac{\kappa(|x_i|)}{|x_i|^2} < 1 \qquad (36)$$

Inverting $\kappa(|x_i|) \le C(|x_i|, \tau_i)$ and $\kappa(|x_i|) \ge D(|x_i|, \tau_i)$ for τ_i, we obtain

$$\begin{aligned}\tau_i \ge \frac{1}{4+3\sqrt{L_m}}\ln\left(1+\frac{4+3\sqrt{L_m}}{2+2\sqrt{L_m}}\frac{\kappa(|x_i|)}{|x_i|^2}\right)\wedge\\ -\frac{1}{4+3\sqrt{L_m}}\ln\left(1-\frac{4+3\sqrt{L_m}}{6+5\sqrt{L_m}}\frac{(|x_i|^2+d_\alpha-\kappa(|x_i|))}{|x_i|^2}\right)\end{aligned} \qquad (37)$$

and using (35)-(36), $\tau_i > 0$. □

The inter-execution times τ_i, $i = 0, 1, \ldots$, may be calculated numerically from (28) or (30) based on the norm of the last-observed state $|x_i|$. For systems where noise does not vanish at the origin, it is likely that $d > 0$ in (12), and, hence, $d_\alpha > 0$. Then the times may be computed from (28), which requires $d_\alpha > 0$. If the constant disturbance is $d = 0$ (which is often the case when $g(0, 0) = 0$), one could compute the inter-execution times from (28) with a $d_\alpha > 0$, although this would only guarantee practical p-th moment stability, and it would introduce a $d > 0$ to (13). To avoid this, for systems where the noise vanishes at the origin, the times can also be computed from (30) with $d_\alpha \ge 0$, leading to p-moment stability.

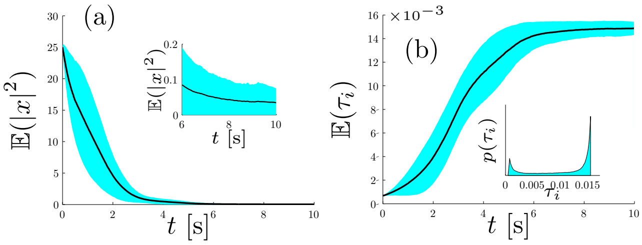

Fig. 1. Linear system from [22] with stochasticity added. (a) Evolution of $\mathbb{E}(|x|^2)$ over a 10 s simulation, averaged over 1000 sample trajectories, with standard deviation bands shown. The initial condition is a vector of magnitude $|x(0)| = 5$ and random direction. The inset shows the end of the simulation in greater detail. (b) Evolution of the mean $\mathbb{E}(\tau_i)$ and (inset) histogram of τ_i. $mean(\tau_i) = 0.0104$ s, $std(\tau_i) = 7.54 \times 10^{-4}$ s, and $\min(\tau_i) = 3.78 \times 10^{-4}$ s.

Note that for some systems, the duration of the inter-execution times τ_i defined by (28) or (30) may be lengthened with increasing d_α, but with this comes a larger value of $\gamma((1 + \theta_d)d)$ in Eq. (6).

4 Numerical Examples

In this section we provide examples and show how the triggering rule should change depending on the form of the intensity of noise at $x = 0$.

4.1 Stochastic Linear System

The first example is drawn from [22]. In the example, we have added Wiener process increments with both a constant scaling factor $\sigma/2$ and a state-dependent coefficient $\sigma_x|x|/2$. The system is

$$\mathrm{d}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-2 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\mathrm{d}t + \begin{bmatrix}0\\1\end{bmatrix}u\mathrm{d}t + \frac{\sigma+\sigma_x|x|}{2}\begin{bmatrix}\mathrm{d}w_1\\\mathrm{d}w_2\end{bmatrix}$$

with $u = x_1 - 4x_2$. Using $V(x) = x^\intercal Px$ as a Lyapunov function, we can obtain $\mathcal{L}V \le -x^\intercal Qx + \sigma_x^2|x|^2 + \sigma^2$ with

$$P = \begin{bmatrix}1 & \frac{1}{4}\\ \frac{1}{4} & 1\end{bmatrix}, \qquad Q = \begin{bmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{3}{2}\end{bmatrix}.$$

Under a sampled-data implementation, this becomes

$$\mathcal{L}V \le -(a - \sigma_x^2)|x|^2 + b|e||x| + \sigma^2$$

where $a = \lambda_m(Q) > 0.44$ is the smallest eigenvalue of Q, and $b = |K^\intercal B^\intercal P + PBK| = 8$. We take the expectation of both sides and apply Hölder's inequality

$$
\begin{aligned}
\mathbb{E}\mathcal{L}V &\le -(a-\sigma_x^2)\mathbb{E}|x|^2 + b\mathbb{E}(|e||x|) + \sigma^2 \\
&\le -(a-\sigma_x^2)\mathbb{E}|x|^2 + b\sqrt{\mathbb{E}(|e|^2)}\sqrt{\mathbb{E}(|x|^2)} + \sigma^2 \\
&\le -(a-\sigma_x^2)\mathbb{E}|x|^2 + b\sqrt{\mathbb{E}(|e|^2)}\sqrt{\mathbb{E}(|x|^2) + \frac{\theta_d}{\theta}\sigma^2} + \sigma^2
\end{aligned}
$$

where the constant term involving $\theta_d > 0$ has been added in order to facilitate the following triggering rule. If we were to assume that

$$\mathbb{E}(|e|^2) \le \theta\mathbb{E}(|x|^2) + \theta_d\sigma^2, \tag{38}$$

for some constant $0 < \theta < (a-\sigma_x^2)^2/b^2$, then $\mathbb{E}\mathcal{L}V \le -(a-\sigma_x^2-b\sqrt{\theta})\mathbb{E}|x|^2 + \sigma^2(1+\frac{\theta_d}{\sqrt{\theta}}b)$, and based on Theorem 1, we will obtain practical stability in the $p=2$ moment, i.e., in the mean square sense. In light of the triggering rule (38), we set $\alpha_v(|x|) = |x|$, and $\lambda_c(|e|) = |e|$ (which are concave and convex, respectively, although not strictly so), meaning that the update times τ_i given $|x_i|$ can be solved numerically using Theorem 3 from

$$A(|x_i|, \tau_i) \le \theta B(|x_i|, \tau_i) + d_\alpha \tag{39}$$

with $d_\alpha = \theta_d\sigma^2/\theta$. Note that with these choices of $\lambda_c(|e|)$ and $\alpha_v(|x|)$, it is possible to write down an analytic triggering rule through an appropriate choice of $\kappa(|x_i|)$ using (33) and (34), but this may decrease the duration of the time between updates.

For our first simulations of this system, we choose $\sigma = \sigma_x = 0.1$. The monotone growth and Lipschitz coefficients in (15) and (16) are

$$
\begin{aligned}
K &= \left(\frac{|BK|^2}{2} + \frac{\sigma_x^2}{4} - 1\right) \vee \frac{|BK|}{2} \vee \frac{\sigma^2}{4} = 7.5025 \\
L_m &= (2\sigma_x^2 + 2|A+BK|^2) \vee 2|BK|^2 = 34,
\end{aligned}
$$

respectively, and $(a-\sigma_x^2)/b = 0.0537$, so we choose $\theta = 0.0028$ and $\theta_d = 0.28$ (so that $d_\alpha = 1$). Fig. 1(a) shows $\mathbb{E}(|x|^2)$ based on 1000 simulations from an initial condition on a random vector of magnitude $|x(0)| = 5$. During simulation, after the state has been measured as x_i, Theorem 3 provides a *deterministic* time $t = t_i + \tau_i$ at which to update the state. However, with each simulation, we obtain different samples x_i, $i = 1, \ldots,$ and, consequently, the values of τ_i are random when not conditioned on x_i. Fig. 1(b) shows the average $\mathbb{E}(\tau_i)$ of these inter-execution times over the 1000 simulations. For comparison, the inter-execution times in [22] range from 0.0058 s to 0.0237 s. As the samples approach the origin, the average inter-execution times $\mathbb{E}(\tau_i)$ increase (Fig. 1(b)), a trend that can be seen in previous deterministic works [5, 22]. With respect to our triggering condition, this is because, for larger $|x_i|$, $A(|x_i|, \tau_i)$ and $C(|x_i|, \tau_i)$ increase faster with τ_i, and $B(|x_i|, \tau_i)$ and

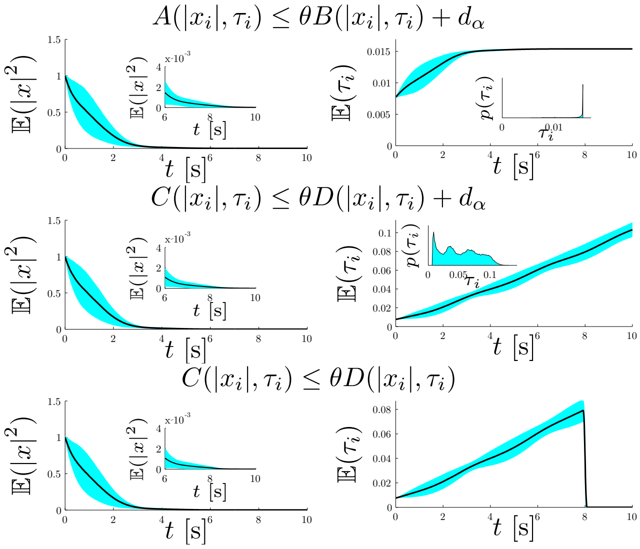

Fig. 2. Linear system from [22] using three different sampling strategies. First row: With $A(|x_i|, \tau_i) \le \theta B(|x_i|, \tau_i) + d_\alpha$, evolution of $\mathbb{E}(|x|^2)$ and $\mathbb{E}(\tau_i)$ for over a 10 s simulation, averaged over 1000 sample trajectories, and with an initial condition of magnitude $|x(0)| = 1$. $mean(\tau_i) = 0.0143$ s, $std(\tau_i) = 6.2 \times 10^{-4}$ s, $\min(\tau_i) = 0.0054$ s. Second row: The same quantities using $C(|x_i|, \tau_i) \le \theta D(|x_i|, \tau_i) + d_\alpha$. $mean(\tau_i) = 0.0511$ s, $std(\tau_i) = 0.002$ s, $\min(\tau_i) = 0.0056$ s. Third row: Initially, the same as the second row, the update rule is changed to a periodic condition ($\tau_i = 3.5 \times 10^{-5}$ s) at $t = 8$ s.

$D(|x_i|, \tau_i)$ will decrease faster. These quantities appear on opposing sides of an inequality in (28) and (30), and consequently, for certain values of θ, these relations will hold for shorter periods of time with larger $|x_i|$.

In the case where $\sigma = 0$ and $\sigma_x = 0.1$, the noise vanishes at the origin, and we can obtain several possible execution rules. The first is (39) with $d_\alpha = 1$, but this only guarantees practical stability in the mean square sense. Since $g(0,0) = 0$, we can also use the execution rule

$$C(|x_i|, \tau_i) \le \theta D(|x_i|, \tau_i) + d_\alpha, \tag{40}$$

which also guarantees practical p-moment stability. However, for this system, we can also set $d_\alpha = 0$ in order to guarantee p-moment stability according to

$$
\begin{aligned}
\mathbb{E}\mathcal{L}V &\le -(a-\sigma_x^2)\mathbb{E}(|x|^2) + b\sqrt{\mathbb{E}(|e|^2)}\sqrt{\mathbb{E}(|x|^2)} \\
&\le -(a-\sigma_x^2-b\sqrt{\theta})\mathbb{E}(|x|^2)
\end{aligned}
$$

Examining the form of $C(|x_i|, \tau_i)$ and $D(|x_i|, \tau_i)$ in (40), note that the triggering rule can be made independent of x_i when $d_\alpha = 0$, and, therefore, it reduces to a periodic

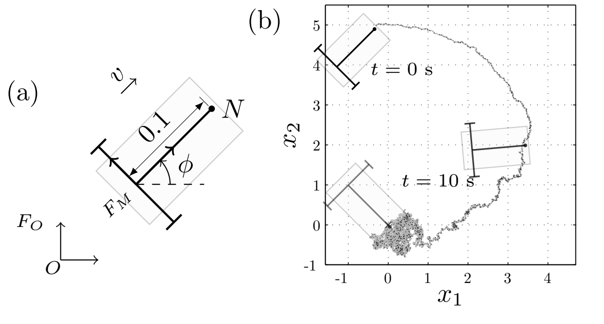

Fig. 3. (a) Diagram of wheeled cart that should drive a point N that is a distance of 0.1 from its wheel axis to the origin O. The coordinates of $\overline{NO}$ in the mobile frame F_M are $[x_1, x_2]^\intercal$. See [20] for details. (b) An example trajectory under a self-triggered implementation (cart drawn not to scale and with arbitrary heading angle).

update rule. In this case, the inter-execution time τ_i are

$$\tau_i = \frac{1}{4\sqrt{L_m} + 3L_m} \times \ln\left(\frac{(1-\theta)^2 + \sqrt{(1-\theta)^2 + 2\theta\frac{6+5\sqrt{L_m}}{1+\sqrt{L_m}}}}{2}\right)$$

Since the periodic update rule may result in shorter inter-execution times, we first apply (40) with $d_\alpha > 0$ (a self-triggered approach) before switching to the periodic rule to ensure asymptotic stability. Fig. 2 shows $\mathbb{E}(|x|^2)$ and $\mathbb{E}(\tau_i)$ based on 1000 simulations using each of these three update rules (with $A(\cdot,\cdot)$, $B(\cdot,\cdot)$, and d_α; $C(\cdot,\cdot)$, $D(\cdot,\cdot)$, and d_α; and the periodic update based on $C(\cdot,\cdot)$ and $D(\cdot,\cdot)$). The second rule (40) results in the longest inter-execution times on average, and it does not appear necessary to switch to the periodic rule in order to achieve asymptotic stability in this example.

4.2 Stochastic Nonlinear Wheeled Cart System

The second example is based on the wheeled cart control problem from [20, Equation (54)]. It describes a wheeled cart that should steer a point on its body to the origin (see Fig. 3(a)) with free final heading angle ϕ. We have added a stochastic disturbance with constant intensity to this example to describe motion in an uncertain environment. The system is described by the equation

$$\mathrm{d}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & x_2 \\ 0 & -(0.1 + x_1) \end{bmatrix} u\mathrm{d}t + \sigma \begin{bmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{bmatrix} \quad (41)$$

where the control $u = [v, \dot{\phi}]^\intercal$ (see Fig. 3(a)). The feedback law $u = [k_1 x_1, k_2 x_2]^\intercal$ with positive gains k_1 and k_2 [20] can be shown to asymptotically stabilize $[x_1, x_2]^\intercal$ to a disc of radius $\sigma/\sqrt{k_1 \wedge 0.1k_2}$. Considering now a sampled-data implementation with $u_1 = k_1(x_1 + e_1)$ and

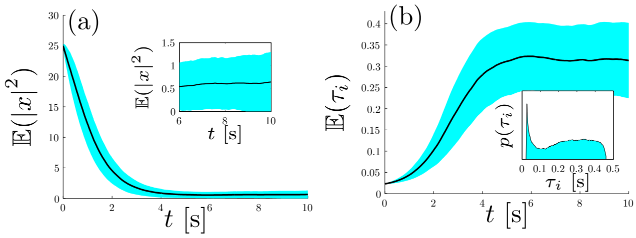

Fig. 4. Nonlinear wheeled cart system from [20] with stochasticity added. (a) Evolution of $\mathbb{E}(|x|^2)$ for 1000 simulations with an initial condition of a random vector of magnitude $|x(0)| = 5$, and with standard deviation bands shown in cyan. The inset shows the end of the simulation in greater detail. (b) Evolution of the mean $\mathbb{E}(\tau_i)$ over the 1000 simulations with standard deviation bands shown. The inset shows a histogram of these times. $mean(\tau_i) = 0.2375$ s, $std(\tau_i) = 0.0282$ s, and $\min(\tau_i) = 0.195$ s.

$u_2 = k_2(x_2 + e_2)$,

$$\mathrm{d}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & x_2 \\ 0 & -(0.1 + x_1) \end{bmatrix} \begin{bmatrix} k_1(x_1 + e_1) \\ k_2(x_2 + e_2) \end{bmatrix} \mathrm{d}t + \sigma \begin{bmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{bmatrix}$$

With the choice of $V(x) = \frac{1}{2}|x|^2$ as a Lyapunov function,

$$\begin{aligned} \mathcal{L}V &= -k_1 x_1^2 - 0.1k_2 x_2^2 - k_1 x_1 e_1 - 0.1k_2 x_2 e_2 + \sigma^2 \\ &\le -\underline{k}|x|^2 + \overline{k}|x||e| + \sigma^2 \end{aligned}$$

where $\underline{k} = k_1 \wedge 0.1k_2$ and $\overline{k} = k_1 \vee 0.1k_2$. Taking the expectation of both sides and applying Hölder's inequality,

$$\begin{aligned} \mathbb{E}\mathcal{L}V &\le -\underline{k}\mathbb{E}(|x|^2) + \overline{k}\mathbb{E}(|x||e|) + \sigma^2 \\ &\le -\underline{k}\mathbb{E}(|x|^2) + \overline{k}\sqrt{\mathbb{E}(|x|^2)}\sqrt{\mathbb{E}(|e|^2)} + \sigma^2 \\ &\le -\underline{k}\mathbb{E}(|x|^2) + \overline{k}\sqrt{\mathbb{E}(|x|^2) + \theta_d\sigma^2}\sqrt{\mathbb{E}(|e|^2)} + \sigma^2 \end{aligned}$$

If we were to assume that

$$\mathbb{E}(|e|^2) \le \theta\mathbb{E}(|x|^2) + \theta_d^2\sigma^2 \quad (42)$$

for $0 < \theta < (\underline{k}/\overline{k})^2$, then $\mathbb{E}\mathcal{L}V \le -(\underline{k} - \overline{k}\sqrt{\theta})\mathbb{E}(|x|^2) + \sigma^2(1 + \frac{\theta_d}{\sqrt{\theta}}\overline{k})$. Using Gronwall's inequality, one can then show that the triggering rule (42) will steer $[x_1, x_2]^\intercal$ to the origin such that for a sufficiently large t, $\mathbb{E}(|x(t)|) \le \mathbb{E}(|x(t)|^2) \le \sigma^2(1 + \theta_d\overline{k}/\sqrt{\theta})/(\underline{k} - \overline{k}\sqrt{\theta})$. Similarly to the previous example, we set $\alpha_v(|x|) = |x|$, and $\lambda_c(|e|) = |e|$ and use the triggering rule (42) with $d_\alpha = \theta_d^2\sigma^2 = 1$. For simulation, we let $k_1 = k_2 = 0.5$, which requires that $0 < \theta < 0.01$, and so we choose $\theta = 0.009$. With $\sigma = 0.4$, the monotone growth coefficient is $K = \overline{k}/2 \vee \sigma^2 = 0.25$. An example trajectory for the resulting sampled-data scheme can be found in Fig. 3(b). Fig. 4 shows the mean square $\mathbb{E}(|x(t)|^2)$ of 1000 trajectories of $x(t)$ starting from a random vector of magnitude $|x(0)| = 5$. The mean inter-execution times $\mathbb{E}(\tau_i)$ again increase as $|x_i|$ approaches the origin (Fig. 4(b)).

5 Conclusions

This paper presents a self-triggered control scheme for state-feedback controlled stochastic differential equations. Since the inequality-based sampling conditions found in previous event- and self-triggered control works may be instantaneously violated in the presence of the stochastic noise considered in this paper, we focus instead on the statistics of the state distribution. These quantities can be predicted based on the last-observed state and are used here to develop a self-triggered control scheme. Since for many systems there is no guarantee that the stochasticity will diminish at the origin, we have considered alongside the error due to sampling a second disturbance caused by non-vanishing noise. We presented triggering conditions based on whether or not this noise vanishes at the origin. The schemes are shown to produce strictly positive inter-execution times that guarantee (practical) p-moment stability of the process.

In future work, elongation of the task periods may be obtainable by taking into account the direction of the error as compared to the current state instead of just its magnitude. Further improvements may be possible by treating the error due to sampling as a delay and more directly applying the stability result from stochastic differential delay systems that is used in this work. The robustness of our scheme to a task delay between state sampling and the control update will be also examined in future work, as well as its application to stochastic problems where control updates or state sampling are expensive or limited, e.g., multi-agent robotic systems.

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