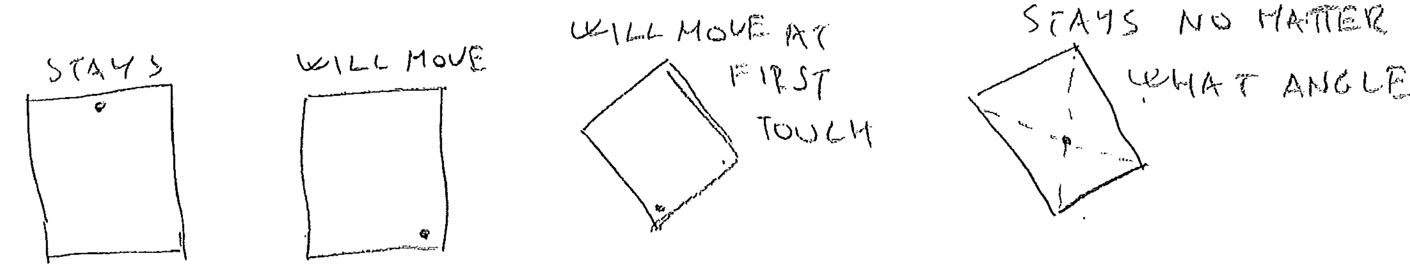
CENTER OF MASS

EVER TRIED PINNING A POSTER TO A WALL?



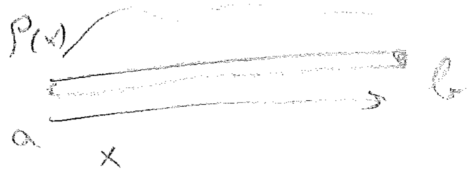
THE LAST POINT IS CALLED THE <u>CENTER OF MASS</u>, AND IT'S A VERY IMPORTANT PHYSICAL OBJECT. FOR MANY PURPOSES (BUT NOT ALL BY ANY MEAN!) WE CAN REDUCE A BODY TO A POINT WITH THE SAME MASS LOCATED IN THE CENTER OF MASS.

IF WE HAVE A FINITE NUMBER OF POINTS IN POSITIONS (x_i, y_i) WITH MASSES m_i THEN THE <u>TOTAL MASS</u> M IS $\sum_{i=1}^{m} m_i$ AND THE CENTER OF MASS $(\hat{x}, \hat{y})$ IS GIVEN BY THE "WEIGHTED AVERAGES" $\hat{x} = \sum_{i=1}^{n} \frac{x_i \cdot m_i}{M}$, $\hat{y} = \sum_{i=1}^{n} \frac{y_i m_i}{M}$

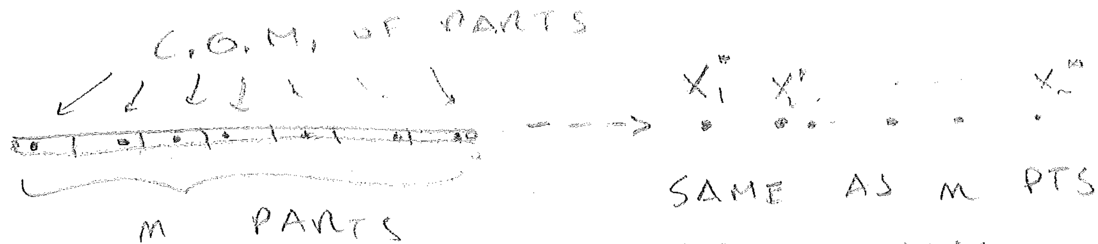
$\sum x_i m_i, \sum y_i m_i$ ARE WRITTEN M_x, M_y AND CALLED "MOMENTS"

EXAMPLE: 1-d OBJECT

ROD WITH DENSITY $\rho(x)$ Kg/m



HOW DO WE FIND C.O.M.?



SAME AS m PTS WITH EQUIV MASS

$$\hat{X} = \sum_{i=1}^{m} \frac{M_i X_i^*}{M} \quad \text{FOR SMALLER AND SMALLER PARTS}$$

$$M_i \sim \underset{\substack{\uparrow \\ \text{DENSITY AT C.O.M.}}}{\rho(X_i^*)} \underset{\substack{\uparrow \\ \text{LENGTH}}}{\Delta_x}$$

$$\hat{X} = \lim_{m \to \infty} \sum_{i=1}^{m} \frac{\rho(X_i^*) X_i^* \Delta_x}{M}$$

$$= \frac{1}{M} \int_a^b \rho(x) \cdot x \, dx =$$

$$= \frac{\int_a^b \rho(x) \cdot x \, dx}{\int_a^b \rho(x) \, dx}$$

EX: BAR 50 cm LONG, $\rho(x) = \frac{1}{100-x}$ gm/cm

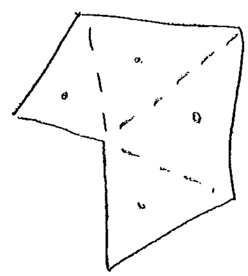
$$\text{MASS} = \int_0^{50} \frac{1}{100-x} dx = -\log(100-x)\Big|_0^{50} = \log(100) - \log(50) = \log 2$$

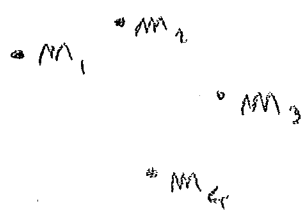
$$\hat{X} = \frac{1}{\log 2} \int_0^{50} \frac{x}{100-x} dx = \frac{1}{\log 2} \int_0^{50} -1 + \frac{100}{100-x} dx = \frac{-50 + 100 \log 2}{\log 2}$$

$$\approx 28.06$$

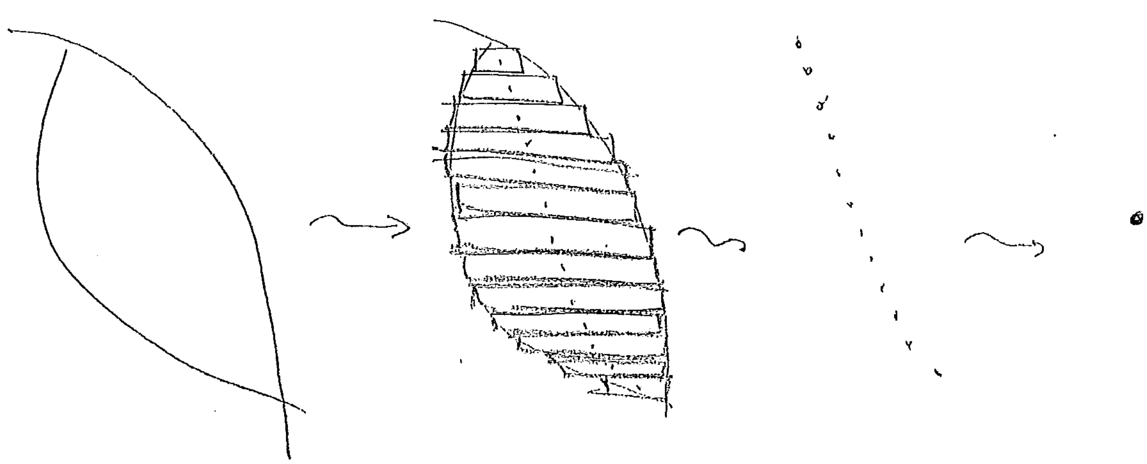
BUT WHAT IF WE HAVE A 2-d OBJECT, SUCH AS A METAL PLATE (of UNIFORM DENSITY ρ)?

IDEA: IF WE SUBDIVIDE AN OBJECT IN m PIECES, THE CENTER OF MASS (C.O.M.) OF THE OBJECT IS THE SAME AS THE C.O.M. OF m POINTS, EACH AT ITS RESPECTIVE PIECE'S C.O.M., EACH HAVING THE MASS OF THE RESPECTIVE PIECE.

SO C.O.M. OF  =

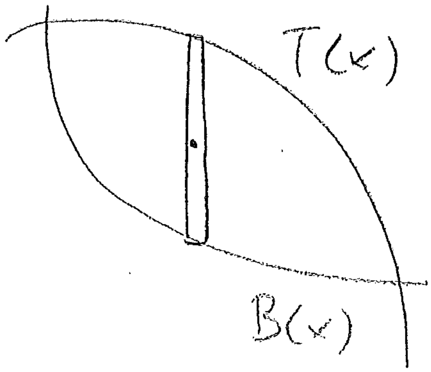
C.O.M. OF 

HOW DO WE USE THIS? WE <u>PANCAKE</u>!



SAY WE HAVE THE REGION

$$\{(x, y) \mid B(x) \leq y \leq T(x),\ a \leq x \leq b\}$$



HEIGHT OF A STRIPE:
$T(x)-B(x)$
WIDTH: Δ_x

C.O.M. OF THE STRIPE

IS AT IS AT $x, \frac{T(x)+B(x)}{2}$

MASS OF STRIPE IS

$\rho \cdot \Delta_x \cdot (T(x)-B(x))$ SO THE y-COORDINATE OF THE C.O.M. IN THE m-STRIPES APPROX IS

$$\frac{1}{M}\sum_{i=1}^{m}(T(x_i)-B(x_i))\cdot\rho\cdot\Delta_x\cdot\frac{T(x)+B(x)}{2} =$$

$$\frac{1}{M}\sum_{i=1}^{m}\frac{\rho(T(x_i)^2-B(x_i)^2)}{2}\cdot\Delta_x$$ WHERE x_i

ARE LEFT ENDP. OR RIGHT ENDP. OR MIDDLE...

GOING TO THE LIMIT $m \to \infty$

WE GET y-COORDINATE OF

C.O.M. = $\hat{y} = \dfrac{1}{2\rho A}\displaystyle\int_a^b \rho\left(T(x)^2 - B(x)^2\right)dx$

$$= \frac{1}{2A}\int_a^b \left(T(x)^2 - B(x)^2\right)dx$$ WHERE

A IS THE AREA OF OUR REGION

HOW ABOUT THE x-COORDINATE?

IT'S EASIER! DOING THE SAME

WE GET

X-COORD = $\hat{x} = \dfrac{1}{A}\displaystyle\int_a^b x\left(T(x) - B(x)\right)dx$

SO IN GENERAL:

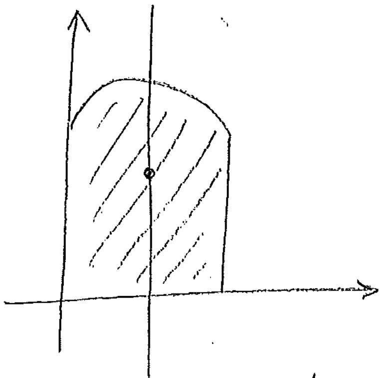
DEF: THE C.O.M. $(\hat{x}, \hat{y})$ OF THE AREA BETWEEN $T(x)$ AND $B(x)$ RUNNING FROM a TO b IS

$$(\hat{x}, \hat{y}) = \left(\frac{1}{A}\int_a^b x\left|T(x) - B(x)\right|dx \,, \; \frac{1}{2A}\int_a^b \left|T(x)^2 - B(x)^2\right|dx \right)$$

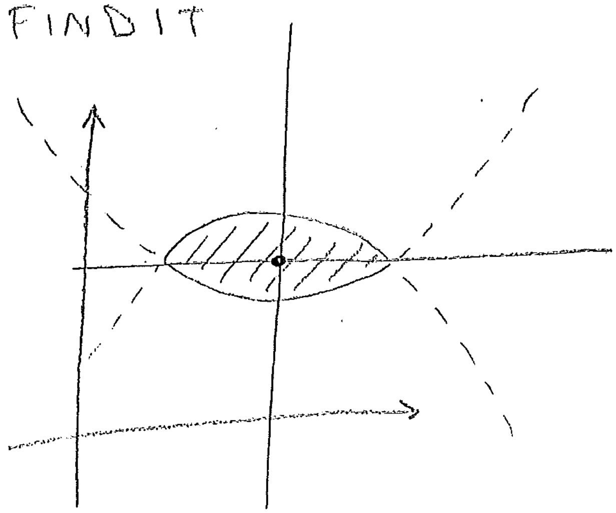
~~WARM-UP~~ EXAMPLES:

1) USE SYMMETRY TO RESTRICT THE POSITION OF THE C.O.M. OF THE FOLLOWING REGIONS TO A LINE, OR TO FIND IT

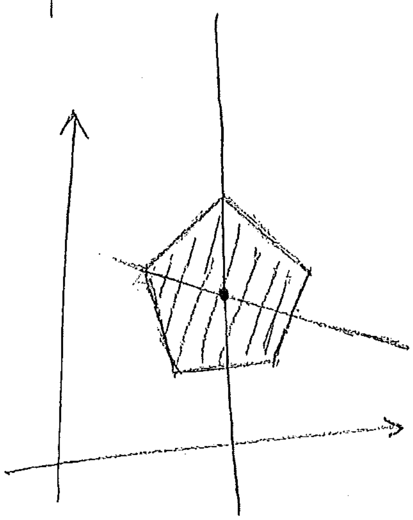
i)



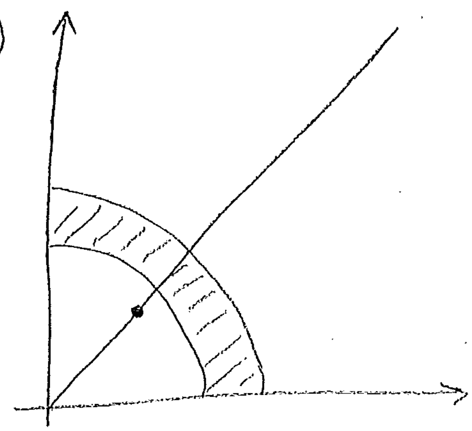
ii)



iii)

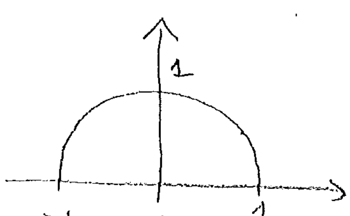


iv)



IDEA: IF A REGION IS SYMMETRIC WITH RESPECT TO A LINE, THE C.O.M. MUST BE ON IT

2) FIND THE C.O.M. OF AN HALF-CIRCLE



$\hat{x} = 0$ BY SYMMETRY

$$\hat{y} = \frac{1}{\pi}\int_{-1}^{1} 1 - x^2 \, dx = \frac{2}{\pi} - \frac{2}{3\pi} = \frac{4}{3\pi}$$