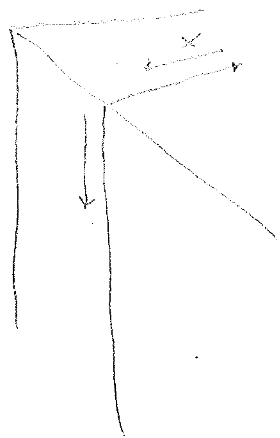
$11X \cdot \frac{27}{63} \Delta x \cdot g$ IF $x < 11$. SO



$$W = g\left(\int_{11}^{63} 11 \cdot \frac{27}{63}\, dx + \int_{0}^{11} x \cdot \frac{27}{63}\right) dx$$

$$= g\left(27 \cdot 11 - 11^2 \cdot \frac{3}{7} + 11^2 \cdot \frac{3}{14}\right) =$$

$$g\left(27 \cdot 11 - 11^2 \cdot \frac{3}{14}\right) \text{ J}$$

EX: A BUCKET WEIGHING 5 N, CONTAINING WATER WEIGHING 2 N IS LIFTED 20 m AT CONSTANT SPEED. IT LEAKS AT A CONSTANT RATE, AND FINISHES DRAINING WHEN IT REACHES THE TOP.

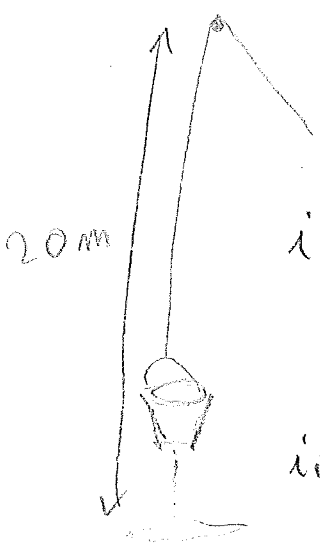
i) WHAT WAS THE WORK SPENT IN LIFTING THE WATER ALONE?

ii) WHAT WAS THE TOTAL WORK?

WEIGHT AT TOP = ∅ N

WEIGHT AT BOTTOM = 2 N

AT HEIGHT $x = -\frac{1}{10} x + 2$

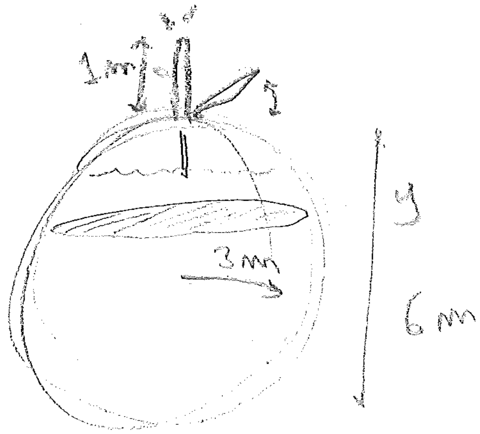


i) TOTAL (WATER) $\displaystyle\int_0^{20} -\frac{1}{10} x + 2\, dx = 40 - \frac{1}{20} x^2 \Big|_0^{20} = 20 \text{ J}$

ON BUCKET $\displaystyle\int_0^{20} 5\, dx = 100 \text{ J}$

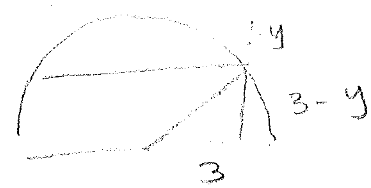
ii) TOTAL 120 J

EXAMPLE: A SPHERICAL TANK OF RADIUS 3 m IS FULL OF WATER (DENSITY $= 10^3$ Kg/m^3) FIND THE ENERGY NEEDED TO PUMP IT OUT FROM THE TOP THROUGH A SPROUT 1 m LONG.



HORIZONTAL SLICE AT DEPTH y:

RADIUS $= \sqrt{9-(3-y)^2} = \sqrt{6y-y^2}$



AREA $= \pi(6y-y^2)$

VOLUME $=$

$$\Delta y\cdot\pi(6y-y^2)$$

MASS $= \Delta y\cdot\pi(6y-y^2)\cdot 10^3$ Kg

WORK ON SLICE : $\underset{\text{DISTANCE}}{(y+1)}\ \underbrace{\Delta y\,\pi(6y-y^2)\cdot 10^3\cdot g}_{\text{FORCE}}$ J

TOTAL WORK:

$$\pi 10^3 g\int_0^6 (y+1)(6y-y^2)\,dy = 10^3 g\pi\int_0^6 -y^3+6y^2+6y\,dx$$

$$= 10^3 g\pi\left(\frac{-6^4}{4}+\frac{5\cdot 6^3}{3}+3\cdot 6^2\right) \approx 4.4\times 10^6 \text{ J}$$

EXAMPLE:

HOOKE'S LAW $\quad F = kx$

K: SPRING CONSTANT
X: AMOUNT OF STRETCHING

A SPRING HAS A NATURAL LENGTH OF 20 CM. A 25 N FORCE IS REQUIRED TO KEEP IT AT 30 CM, FIND WORK TO STRETCH FROM 20 TO 25 CM.

i) WE FIND K. $\quad k(0.3-0.2) = 25$

$0.1k = 25 \quad k = 250 \; N/m$

ii) $$\int_0^{0.05} F(x)\,dx = \int_0^{0.05} 250x\,dx = 125x^2\Big|_0^{0.05} =$$

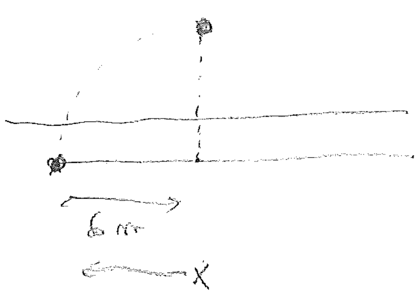
$125 \times 0.0025 = 0.3125 \; J$

EXAMPLE: A CHAIN LYING ON THE GROUND IS 10 m LONG AND HAS CONSTANT DENSITY OF 8 kg/m. FIND WORK TO RAISE ONE END TO 6 m.

Δ_x CHAIN AT x MOVES UP x METRES.

FORCE ON Δ_x CHAIN: $\quad g \cdot 8 \cdot \Delta_x$

WORK ON Δ_x CHAIN: $\quad x \cdot g \cdot 8 \Delta_x$



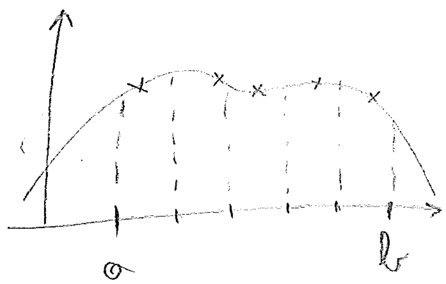
TOTAL: $$\int_0^6 8gx\,dx = 4gx^2\Big|_0^6 = 144g$$

$\approx 1411.2 \; J$

OR $x \to$ $\int_0^6 8g(6-x)\,dx$, SAME RESULT.

AVERAGES

THE AVERAGE OF n DATA POINTS $F_1, \dots, F_n$ IS $\frac{1}{n}(F_1 + \dots + F_n)$. WHAT IF WANTED THE AVERAGE OF A FUNCTION $f(x)$ ON $[a,b]$?



WE CAN TRY THIS. DIVIDE $[a,b]$ IN n INTERVALS, THEN PICK RANDOM PTS x_1^* IN $[x_0, x_1]$, ..., x_n^* IN $[x_{n-1}, x_n]$, AND TAKE $\frac{1}{n}\left(f(x_1^*) + \dots + f(x_n^*)\right)$. NOW WE LET $n \to \infty$. WHAT HAPPENS?

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{f(x_i^*)}{n} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} \cdot \frac{1}{b-a}$$

$$= \lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

DEF: $\mathrm{AVG}_{[a,b]} f(x) = \frac{1}{b-a} \int_a^b f(x)\,dx$.

EXAMPLE: THE AVG OF $f(x) = x^2$ ON $[0,3]$ IS

$$\frac{1}{3}\int_0^3 x^2\,dx = \frac{1}{9}x^3\Big|_0^3 = 3$$

EXAMPLE: A PARTICLE MOVES WITH SPEED $v(t)$ ON TIME INTERVAL $[0,T]$

AVERAGE SPEED IS

$$\frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{T}\underset{\substack{\uparrow\\ \text{POSITION}}}{X(t)}\Big|_0^T = \frac{X(T)-X(0)}{T}$$

$$= \frac{\text{DISTANCE}}{\text{TIME}}$$

EXAMPLE: CUP OF COFFEE HAS TEMP $95^\circ C$ IN A $20^\circ C$ ROOM. BY NEWTON'S LAW OF COOLING AFTER t MINS TEMP IS

$$T(t) = 20 + 75e^{-t/50}$$

AVG TEMP IN FIRST HALF HOUR

$$\frac{1}{30}\int_0^{30} T(t)\,dt = \frac{1}{30}\int_0^{30} 20 + 75e^{-t/50}\,dt =$$

$$20 - \frac{75\cdot 50}{30}\left(e^{-30/50} - 1\right) \cong 76.4^\circ C$$