COMPARISONS

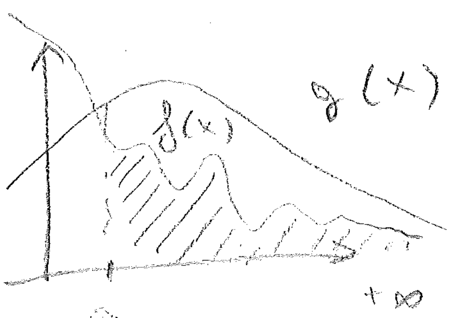
THM (COMPARISON TEST): a, b IN $\mathbb{R} \cup \pm\infty$.

i) SUPPOSE THAT

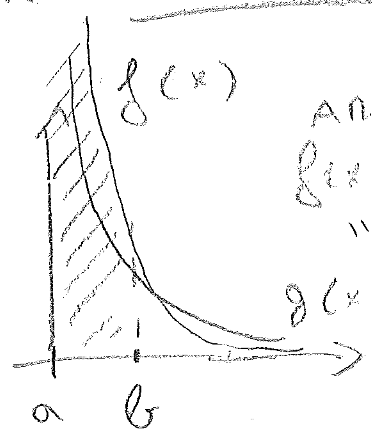
$$0 \le f(x) \le g(x) \quad \text{OR} \quad g(x) \le f(x) \le 0$$

ON (a,b). THEN IF $\int_a^b g(x)\,dx$ CONVERGES $\int_a^b f(x)\,dx$ CONVERGES.

ii) SUPPOSE THAT $0 \le g(x) \le f(x)$ OR $f(x) \le g(x) \le 0$. THEN IF $\int_a^b g(x)\,dx$ DIVERGES $\int_a^b f(x)\,dx$ DIVERGES.

i) 

AREA UNDER $f(x) \le$ AREA UNDER $g(x)$

ii 

AREA UNDER $f(x) \ge$ AREA UNDER $g(x)$.

EXAMPLE:

$\int_1^\infty \frac{1}{x^2+x}\,dx$ CONVERGES BY COMPARISON WITH

$\int_1^\infty \frac{1}{x^2}\,dx$ AS $\frac{1}{x^2+x} < \frac{1}{x^2}$ FOR $x \ge 1$

EXAMPLE: DOES

$$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx \quad \text{CONVERGE?}$$

IDEA: $\sin^2 x \le 1$

$\sin^2 x \le 1$ SO $\frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ SO

CONV BY COMPARISON WITH $\frac{1}{x^2}$

EXAMPLE: DOES

$$\int_3^\infty \frac{1}{\sqrt{x^2-5}}\,dx$$

CONVERGE?

IDEA: $\sqrt{x^2-5} \approx x$ FOR BIG x

$$\frac{1}{\sqrt{x^2-5}} \ge \frac{1}{\sqrt{x^2}} = \frac{1}{x}$$

SO IT DIVERGES BY COMPARISON WITH $\frac{1}{x}$

EXAMPLE: DOES

$$\int_0^3 \frac{1}{x^3+x^2+x^{\frac{1}{2}}}\,dx \quad \text{CONVERGE?}$$

IDEA: $x^3+x^2+x^{\frac{1}{2}} \approx x^{\frac{1}{2}}$ FOR x SMALL

SINCE $\to$ $x^3+x^2+x^{\frac{1}{2}} \ge x^{\frac{1}{2}}$ SO IT CONV BY COMP WITH $\frac{1}{x^{\frac{1}{2}}}$

EXAMPLE: DOES

$$\int_2^{\infty} \frac{1}{x^3 - x}\,dx \text{ CONV?}$$

IDEA: $x^3 - x \approx x^3$ FOR LARGE x

HOW TO SOLVE FORMALLY:

$x^3 > 2x$ FOR $x \geq 2$ (WHY? TRUE FOR $x = 2$, $(x^3 - 2x)' = 3x^2 - 2 > 0$ FOR $x \geq 2$)

SO $\frac{x^3}{2} > x$ SO $x^3 - x > \frac{x^3}{2}$ SO

$$\frac{1}{x^3 - x} < \frac{2}{x^3}$$ SO IT CONVERGES.

EXAMPLE: DOES

SING → $$\int_{\frac{1}{2}}^{1} \frac{1}{x^3 - x}\,dx \text{ CONV?}$$

IDEA: $x^3 - x = x(x-1)(x+1)$

CLOSE TO $x = 1$

$x^3 - x \approx (x-1) \cdot 2$ (DIV!)

HOW TO SOLVE FORMALLY

FOR $\frac{1}{2} \geq x \geq 1$ $\frac{3}{4} \geq x(x+1) \geq 2$

SO $|x^3 - x| \geq \frac{3}{4}\, x - 1$ SO DIVERGES

BY COMPARISON WITH $\frac{4}{3} \cdot \frac{1}{x-1}$

WORK

WORK IS THE AMOUNT OF ENERGY IMPARTED ON A BODY BY A FORCE WHILE MOVING IT FROM a TO b. IT CAN BE SEEN AS THE AMOUNT OF ENERGY NEEDED TO MOVE AN OBJECT FROM a TO b AGAINST A FORCE (E.G. GRAVITY, A SPRING, A MAGNET)

IT'S DEFINED AS

$$W = \int_a^b F(x)\,dx$$

IN PARTICULAR, IF F IS CONSTANT, $W = F(b-a)$.

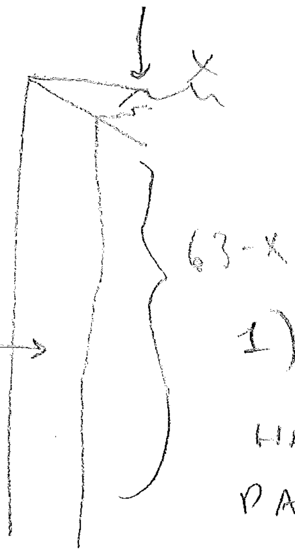
HOW DO WE COMPUTE IT?

WE PANCAKE OUR "PROBLEM" IN SLICES FOR WHICH THE FORCE IS CONSTANT, THEN INTEGRATE OVER THEM.

THE BEST WAY TO UNDERSTAND THIS IS BY (NUMEROUS) EXAMPLES.

EXAMPLE: A CHAIN 63 m LONG, OF MASS = 27 Kg HANGS OVER THE LEDGE OF A TALL BUILDING. FIND WORK REQUIRED TO LIFT 11 m OF CHAIN TO THE TOP.

PART ON TOP (NO FORCE ACTING)



PART HANGING $F=mg$ ACTING

TWO WAYS:

1) (BEST) CONSIDER WHOLE HANGING PART, INTEGRATE OVER PART PULLED

x = PART ON TOP $\quad 63-x$ = PART HANGING

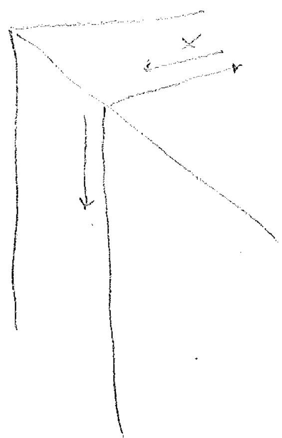
$$F = \underbrace{(63-x)}_{\text{LENGHT}} \cdot \underbrace{\frac{27}{63}}_{\text{MASS}/m} g$$

$$W = g\int_0^{11} 27 - \frac{3}{7}x \, dx = g\left(27\cdot 11 - \frac{3}{14}\cdot 11^2\right) \simeq$$

$$9.8\left(27\cdot 11 - \frac{3}{14}\cdot 11^2\right) = 2656.5 \text{ JOULE}$$

2) PANCAKE CHAIN. Δ_x OF CHAIN POSITION x MOVES 11 IF $x \geq 11$, x IF $x < 11$. MASS OF Δ_x CHAIN $= \frac{27}{63}\Delta_x$ Kg, SO WE GET WORK ON Δ_x CHAIN $= 11 \cdot \frac{27}{63}\Delta_x g$ IF $x \geq 11$, AND

$11x \cdot \frac{27}{63} \Delta x \cdot g$ IF $x < 11$. SO



$$W = g\left(\int_{11}^{63} \frac{11 \cdot 27}{63}\, dx + \int_{0}^{11} x \cdot \frac{27}{63}\right) dx$$

$$= g\left(27 \cdot 11 - 11^2 \cdot \frac{3}{7} + 11^2 \cdot \frac{3}{14}\right) =$$

$$g\left(27 \cdot 11 - 11^2 \cdot \frac{3}{14}\right) \text{ J}$$

EX: A BUCKET WEIGHING 5N, CONTAINING WATER WEIGHING 2N IS LIFTED 20 m AT CONSTANT SPEED. IT LEAKS AT A CONSTANT RATE, AND FINISHES DRAINING WHEN IT REACHES THE TOP.

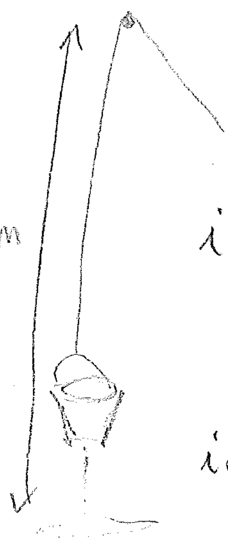
i) WHAT WAS THE WORK SPENT IN LIFTING THE WATER ALONE?

ii) WHAT WAS THE TOTAL WORK?

WEIGHT AT TOP $= \varnothing$ N

WEIGHT AT BOTTOM $= 2$ N

AT HEIGHT x $= -\frac{1}{10}x + 2$



20m

i) TOTAL (WATER) $\int_0^{20} -\frac{1}{10}x + 2\, dx = 40 - \frac{1}{20}x^2 \Big|_0^{20} = 20$ J

ON BUCKET $\int_0^{20} 5\, dx = 100$ J

ii) TOTAL 120 J