EX: $\int_{-\pi/2}^{\pi/2} \cos(x)\,dx$, 4 STEPS

- THE X-VALUES: $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$ $\Delta_x = \frac{\pi}{4}$

SO X VALUES

$-\frac{3}{8}\pi, -\frac{\pi}{8}, \frac{\pi}{8}, \frac{3}{8}\pi$

- $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx \approx \left[\cos\left(-\frac{3}{8}\pi\right) + \cos\left(-\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{3}{8}\pi\right)\right] \cdot \frac{\pi}{4}$

$= 2.05234\ldots$

EXACT VALUE IS $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx = 2$

SO ABSOLUTE ERROR $|2 - 2.05234\ldots| \approx 0.052$

RELATIVE ERROR $\frac{0.052\ldots}{2} \approx 0.026$

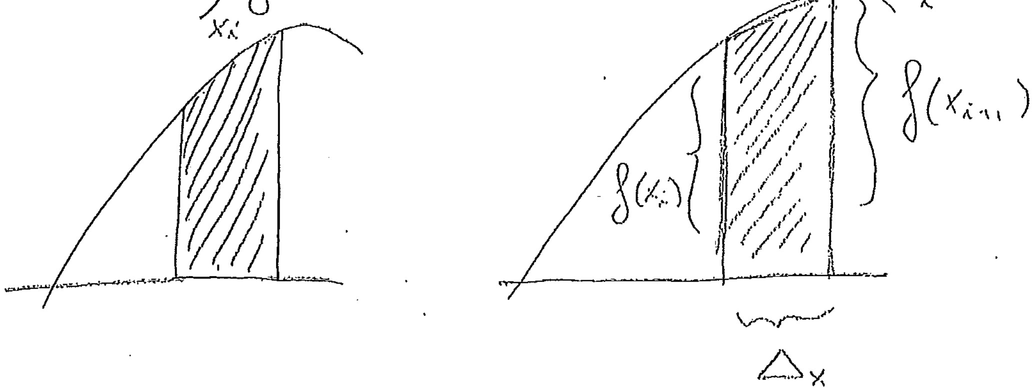
PERCENTAGE ERROR $\approx 2.6\%$

LATER WE'LL FIND <u>A PRIORI</u> BOUNDS FOR ALL THESE ERRORS

TRAPEZOIDAL RULE

WE APPROXIMATE THE FUNCTION BETWEEN x_i AND x_{i+1} WITH THE LINEAR FUNCTION $L(x)$ SUCH THAT $L(x_i) = f(x_i)$ AND $L(x_{i+1}) = f(x_{i+1})$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \qquad\qquad \int_{x_i}^{x_{i+1}} L(x)\,dx = \left(\frac{f(x_i)+f(x_{i+1})}{2}\right)\cdot\Delta_x$$



DEF: SET $\Delta_x = \frac{b-a}{n}$. THE TRAPEZOIDAL APPROXIMATION (WITH n STEPS) IS

$$\int_a^b f(x)\,dx \approx \left[\left(\frac{f(a)+f(a+\Delta_x)}{2}\right) + \left(\frac{f(a+\Delta_x)+f(a+2\Delta_x)}{2}\right) + \dots + \left(\frac{f(a+(n-1)\Delta_x)+f(b)}{2}\right)\right]\cdot\Delta_x = \left[f(a)+2f(a+\Delta_x)+2f(a+2\Delta_x) + \dots + 2f(a+(n-1)\Delta_x)+f(b)\right]\cdot\frac{\Delta_x}{2} =$$

$$\left[\frac{f(a)}{2} + \sum_{i=1}^{n-1} f(a+i\Delta_x) + \frac{f(b)}{2}\right]\cdot\Delta_x$$

EX: APPROXIMATE $\int_0^1 \frac{4x}{1+x^2}\,dx$ WITH THE TRAPEZOIDAL RULE, $n=4$

- WE HAVE $\Delta_x = \frac{1}{4}$, SO THE X-VALUES WILL BE $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

- THE APPROXIMATION IS

$$\int_0^1 \frac{4x}{1+x^2}dx \approx \left[\frac{0}{2} + \frac{1}{1+\frac{1}{16}} + \frac{2}{1+\frac{1}{4}} + \frac{3}{1+\frac{9}{16}} + \frac{4}{2}\cdot\frac{1}{2} \right]\cdot\frac{1}{4}$$

$= 1.365294\ldots$

OUR ERROR IS:

ABSOLUTE $|\log(4) - 1.365294\ldots| \doteq 0.02100\ldots$

RELATIVE $\dfrac{0.02100}{\log(4)} \approx 0.0151$

PERCENTAGE 1.5%

EX: APPROXIMATE $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx$ WITH TRAPEZOID RULE, $m = 4$

- $\Delta x = \frac{\pi}{4}$, X VAL $-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}$

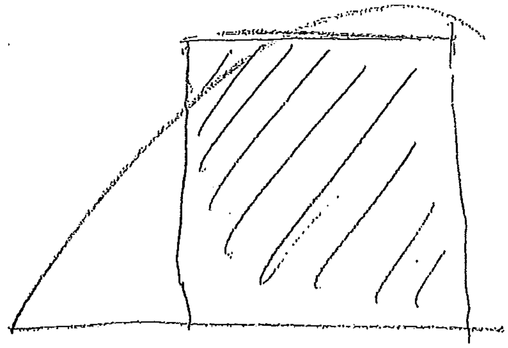
- $$\int_0^1 \cos(x)\,dx \approx \left[\frac{\cos(-\frac{\pi}{2})}{2} + \cos(-\frac{\pi}{4}) + \cos(0) + \cos(\frac{\pi}{4}) + \frac{\cos(\frac{\pi}{2})}{2} \right]\cdot\frac{\pi}{4} = (1+\sqrt{2})\cdot\frac{\pi}{4} \doteq 1.896118\ldots$$

ABSOLUTE ERROR $|2 - (1+\sqrt{2})\cdot\frac{\pi}{4}| = 0.103\ldots$

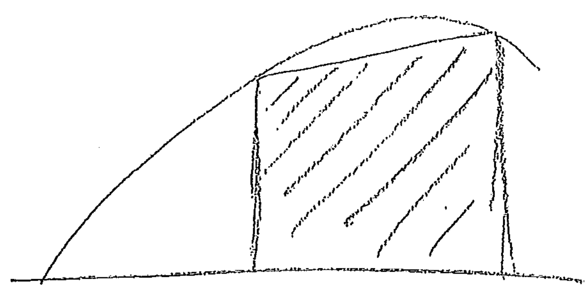
RELATIVE $\dfrac{0.103\ldots}{2} \approx 0.051$ PERCENTAGE 5.1%

OPTIONAL

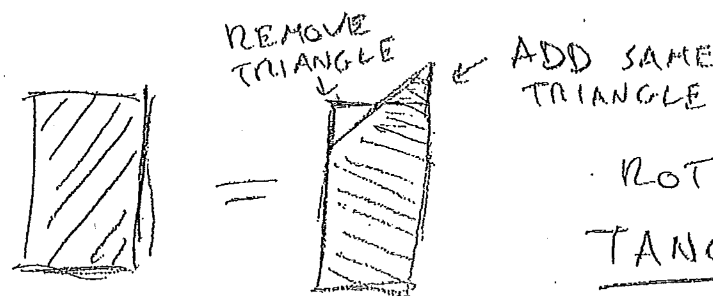
WHY DOES THE MIDPOINT RULE SEEM BETTER THAN TRAPEZOIDAL? IT SEEMS THAT THIS



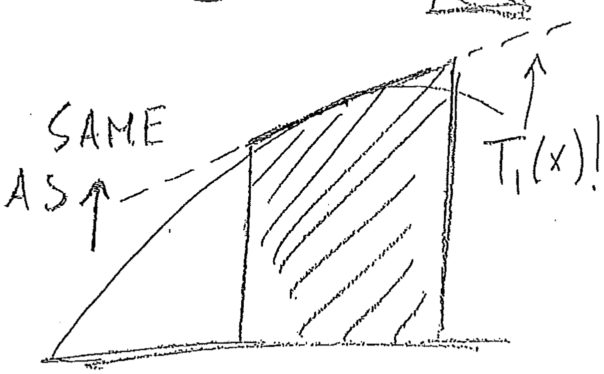
SHOULD BE LESS SMART THAN THIS!



IDEA: WITHOUT CHANGING THE AREA, WE CAN ROTATE THE TOP SEGMENT IN THE MIDPOINT RULE!



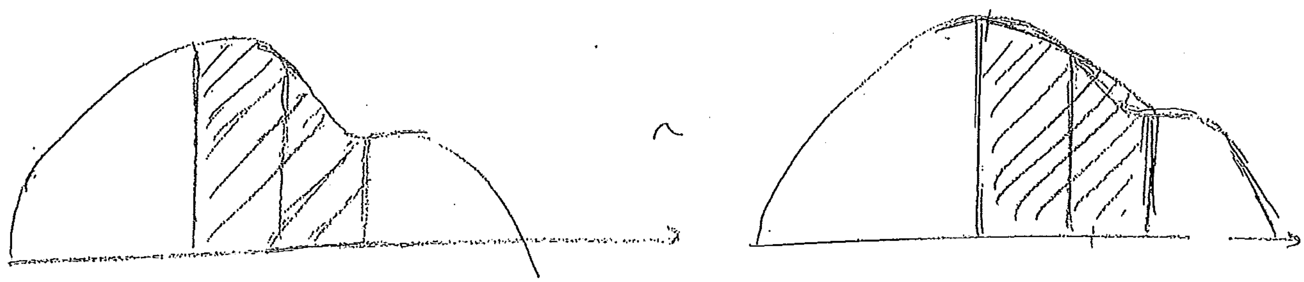
SO WE CAN ROTATE IT TO BE TANGENT TO $f(x)$!



IT'S THE LINEAR APPROXIMATION OF $f(x)$ AT THE MIDPOINT!!

SIMPSON'S RULE

ROUGHLY SPEAKING, SIMPSON'S RULE APPROXIMATES $f(x)$ ON TWO ADJACENT INTERVALS WITH A QUADRIC



DEF: SET $\Delta_x = \frac{b-a}{m}$, AND SUPPOSE m IS EVEN. THE SIMPSON'S RULE APPROX IS

$$\int_a^b f(x)\,dx \approx \Big[f(a) + 4f(a+\Delta_x) + 2f(a+2\Delta_x) + \dots +$$

$$+ 2f(a+(m-2)\Delta_x) + 4f(a+(m-1)\Delta_x) + f(b) \Big] \frac{\Delta_x}{3}$$

$$= \Big[f(a) + f(b) + 4\sum_{\substack{i \text{ ODD} \\ 1 \le i \le m-1}} f(a+i\Delta_x) + 2\sum_{\substack{j \text{ EVEN} \\ 2 \le i \le m-2}} f(a+j\Delta_x) \Big] \frac{\Delta_x}{3}$$

EX: $\int_0^1 \frac{4x}{1+x^2}\,dx$ USING SIMPSON'S, $m=4$

- VALUES OF x: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$
- APPROX $\int_0^1 \frac{4x}{1+x^2}\,dx = \Big[f(0) + f(1) + 4\big(f(\tfrac{1}{4}) + f(\tfrac{3}{4})\big) + 2f(\tfrac{1}{2}) \Big] \frac{1}{12}$

$$= \Big[0 + 2 + 4\Big(\frac{1}{1+\frac{1}{16}} + \frac{3}{1+\frac{9}{16}}\Big) + 2\Big(\frac{2}{1+\frac{1}{4}}\Big) \Big] \cdot \frac{1}{12} = \frac{1179}{850} = 1.387058\dots$$

- ERROR:

ABSOLUTE $\left| \log(4) - 1.387058... \right| = 0.00076...$

RELATIVE $\dfrac{0.00076...}{\log(4)} = 0.000551... \approx 0.0005$

PERCENTAGE $0.000551... \cdot 100 \approx 0.05\%$

EX: $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\, dx$ WITH SIMPSON'S, $n=4$

- X VALUES: $-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{4}$

- APPROXIMATION $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\, dx \approx$

$$\left[\cos\left(-\tfrac{\pi}{2}\right) + \cos\left(\tfrac{\pi}{2}\right) + 4\left(\cos\left(-\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\right) + 2\cos(0)\right] \frac{\pi}{12}$$

$$= \left[0 + 0 + 4\left(\sqrt{2}\right) + 2\right] \frac{\pi}{12} = \left(2\sqrt{2}+1\right)\frac{\pi}{6} \approx 2.00455...$$

ERROR:

ABSOLUTE $\left| 2 - \left(2\sqrt{2}+1\right)\frac{\pi}{6} \right| = 0.00455... \approx 0.0045$

RELATIVE $\dfrac{0.00455...}{2} = 0.00277... \approx 0.0027$

PERCENTAGE $\approx 0.0027 \times 100 = 0.27\%$

ERROR BOUNDS

WE NEED TO BE ABLE TO <u>PREDICT</u> THE <u>WORST CASE</u> ERROR.

<u>THM</u>:

ASSUME THAT $|f''(x)| \leq M$ FOR ALL x BETWEEN a AND b. THEN

- THE ABSOLUTE ERROR WHEN APPROXIMATING $\int_a^b f(x)\,dx$ WITH THE MID. RULE IS BOUNDED BY $\frac{M}{24}\frac{|b-a|^3}{n^2}$

- THE ABSOLUTE ERROR WHEN APPROXIMATING $\int_a^b f(x)\,dx$ WITH THE TRAPEZOIDAL RULE IS BOUNDED BY $\frac{M}{12}\frac{|b-a|^3}{n^2}$

NOW ASSUME $|f^{(4)}(x)| \leq L$ FOR ALL x BETWEEN a AND b. THEN

- THE ABSOLUTE ERROR WHEN APPROXIMATING $\int_a^b f(x)\,dx$ WITH THE SIMPSON'S RULE IS BOUNDED BY $\frac{L}{180}\frac{(b-a)^5}{n^4}$

REMARK: THESE SHOW THAT IF WE INCREASE THE NUMBER OF STEPS BY TEN TIMES, THE MIDPOINT AND TRAPEZOIDAL PRECISION INCREASES BY A FACTOR $10^2 = 100$, WHILE SIMPSON'S PRECISION INCREASES BY A FACTOR $10^4 = 10{,}000$!

EX: WHAT'S THE ACCURACY FOR THE MIDPOINT APPROX ON $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx$?

$$f(x) = \cos(x), \quad f''(x) = -\cos(x), \quad |f''(x)| \leq 1$$

SO WITH $M = 1$, MAX ERROR

$$\frac{1}{24} \frac{\left|\frac{\pi}{2} + \frac{\pi}{2}\right|^3}{n^2} = \frac{\pi^3}{24} \cdot \frac{1}{n^2} \simeq \frac{1{,}29}{n^2}$$

EX: WHAT'S THE ACCURACY FOR THE TRAPEZOIDAL APPROX ON $\int_0^1 e^{-x^2}\,dx$?

$$\left(e^{-x^2}\right)'' = -2e^{-x^2} + 4x^2 e^{-x^2} = 2(2x^2 - 1)e^{-x^2}$$

LET' BOUND EACH FACTOR

$4x^2 - 2$ IS INCREASING ON $[0,1]$, SO MAX/MIN AT ENDPOINTS $4 \cdot 0^2 - 2 = -2$, $4 \cdot 1^2 - 2 = 2$

SO $|4x^2 - 2| \leq 2$ ON $[0,1]$

$e^{-x^2} \leq 1$ ON $[0,1]$ SO $\underline{M = 2}$

THEN WE GET

$$\text{ABSOLUTE ERROR} \leq \frac{1}{6}\frac{|1-0|^3}{m^2} = \frac{1}{6m^2}$$

EX: HOW MANY STEPS DO WE NEED TO GET AN ACCURACY OF 10^{-6} IN PREVIOUS CASE?

WE WANT $\frac{1}{6m^2} \leq \frac{1}{10^6} \sim 10^6 \leq 6m^2$

$$m \geq \sqrt{\frac{10^6}{6}} = \frac{10^3}{\sqrt{6}} \approx 409$$

EX: WHAT IF WE USE MIDPOINT?

SAME BUT THE FORMULA IS

$$\frac{M}{24}\frac{|b-a|^3}{m^2} = \frac{1}{12}\cdot\frac{1}{m^2} \quad \text{SO} \quad m \geq \sqrt{\frac{10^6}{12}} = \frac{10^3}{2\sqrt{3}} \approx 289$$

EX: WHAT IF WE USE SIMPSONS?

$$\frac{d^4}{dx} e^{-x^2} = 4(4x^4 - 12x^2 + 3)e^{-x^2}$$

NOW $e^{-x^2} \leq 1$. LET'S SPLIT THE + AND − PARTS OF $4x^4 - 12x^2 + 3$

- $3 \leq 4x^4 + 3 \leq 7$ ON $[0,1]$. • $-12 \leq 12x^2 \leq 0$ ON $[0,1]$. SO $-9 \leq 4x^4 - 12x^2 + 3 \leq 7$

$M = 4 \cdot 9 \cdot 1 = 36$

THE ERROR IS THEN

$$\frac{36}{180}\ \frac{(1-0)^5}{m^4} = \frac{1}{5\,m^4}$$

So $\quad 5m^4 \geq 10^6 \quad m^4 \geq 2.10^5 = 20.000$

$m \geq \sqrt[4]{20,000} = 10\sqrt[4]{20} = 21.1474\ldots$

$m \geq 22$.