WARM-UP

- COMPUTE $\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\,dx$ (USING TRIG SUBS)
- COMPUTE $\int \sec x\,dx$ (USING PARTIAL FRAC)

- WE WANT TO REWRITE x^2-2x-3 IN A STANDARD FORM $y^2+b \leftarrow$ CAN BE <0

$y = x-a$

$$(x-a)^2+b = x^2-2x-3$$

$$x^2 \underbrace{-2ax}_{} + a^2 + b = x^2 \underbrace{-2x}_{} - 3$$

$a=1$

$$\cancel{x^2} - 2\cancel{x} + 1 + b = \cancel{x^2} - 2\cancel{x} - 3 \sim b = -4$$

$$x^2-2x-3 = (x-1)^2-4$$

$$\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\,dx = \int_3^5 \frac{\sqrt{(x-1)^2-4}}{x-1}\,dx = \int_2^4 \frac{\sqrt{y^2-4}}{y}\,dy$$

$\boxed{y = 2\sec u}$ $= \int_{0=\operatorname{arcsec} 1}^{\frac{\pi}{3}=\operatorname{arcsec} 2} \frac{2\sqrt{\sec^2 u-1}}{\cancel{\sec u}} \cancel{\sec u}\tan u\,du =$

$u = \operatorname{arcsec}\frac{y}{2}$

$$\int_0^{\frac{\pi}{3}} 2\tan^2 u = 2\int_0^{\frac{\pi}{3}} \sec^2 u - 1\,du = 2\tan u - u \Big|_0^{\frac{\pi}{3}} = 2\sqrt{3} - \frac{2}{3}\pi$$

- $\sec x = \dfrac{1}{\cos x} = \dfrac{\cos x}{\cos^2 x} = \dfrac{\cos x}{1-\sin^2 x}$

$$\int \sec x \, dx = \int \frac{\cos x}{1-\sin^2 x}\, dx \underset{u=\sin x}{=} \int \frac{1}{1-u^2}\, du \Big|_{u=\sin x}$$

$$= -\int \frac{1}{u^2-1}\, du = \ldots$$

TO NOT GET CONFUSED WITH ± 1

$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)}$$

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u-1)$$

$u=1 \sim 1 = 2A \qquad A = \frac{1}{2}$

$u=-1 \sim 1 = -2B \qquad B = -\frac{1}{2}$

$$= -\frac{1}{2}\int \frac{1}{u-1} - \frac{1}{u+1}\, du = -\frac{1}{2}\left(\log|u-1| - \log|u+1|\right) + c$$

$$= -\frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + c = -\frac{1}{2}\log\left|\frac{\sin x - 1}{\sin x + 1}\right| + c$$

PARTIAL FRACTIONS

SO FAR WE CAN SOLVE THE INTEGRAL

$$\int \frac{f(x)}{g(x)}\,dx \quad \text{WHEN } f(x), g(x) \text{ POLYNOMIALS}$$

IF $g(x)$ FACTORS AS A PRODUCT OF POWERS OF LINEAR FACTORS $(x-a_i)$

EX: $\displaystyle\int \frac{x^3}{x^3-3x^2+4}\,dx$

0) LONG DIVISION:

$$\begin{array}{c|c} x^3 & x^3 - 3x^2 + 4 \\ \hline 3x^2 - 4 & 1 \end{array}$$

$$x^3 = (x^3-3x+4) + 3x^2 - 4$$

$$\frac{x^3}{x^3-3x+4} = 1 + \frac{3x^2-4}{x^3-3x+4}$$

1) FACTORIZE $x^3 - 3x^2 + 4 \;\; \pm 2, \pm 1 = (x^2-4x-4)(x+1)$

LOOK FOR DIVISORS

$= (x-2)^2(x+1)$

2) $$\frac{3x^2-4}{(x-2)^2(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+1)}$$

$$3x^2 - 4 = A(x-2)(x+1) + B(x+1) + C(x-2)^2$$

$x=2 \quad 8 = 3B \qquad x=-1 \quad -1 = 9C \qquad C = \frac{-1}{9}$

$$3x^2 = Ax^2 + Cx^2 = Ax^2 + \frac{x^2}{9}$$

$$\frac{28}{9}x^2 = Ax^2 \qquad A = \frac{28}{9}$$

3) $$\int 1 + \frac{8}{3}\frac{1}{(x-2)^2} - \frac{1}{9(x+1)} + \frac{28}{9}\frac{1}{(x-2)}\,dx =$$

$$x - \frac{8}{3}\frac{1}{x-2} - \frac{\log|x+1|}{9} + \frac{28}{9}\log|x-2| + C$$

WHAT ABOUT IRREDUCIBLE QUADRATIC?

Q: CAN WE $\int dx \quad \frac{1}{x^2+b^2}, \quad \frac{x}{x^2+b^2}$?

A: $$\int \frac{1}{x^2+b^2}\,dx = \frac{\arctan \frac{x}{b}}{b}$$

$$\int \frac{x}{x^2+b^2}\,dx = \frac{\log(x^2+b^2)}{2}$$

SO WE CAN INTEGRATE SOME IRREDUCIBLE QUADRATIC FACTORS... OR ALL?

Ex: $$\int \frac{x+1}{x^2-2x+5}\,dx$$ $4 - 20 = -16 < 0$ SO CPLX ROOTS

WE WANT $x^2 - 2x + 5 = (x-a)^2 + b$

$$x^2 - 2x + 5 = x^2 - 2ax + a^2 + b$$

$a = 1 \qquad b = 5 - a^2 = 4 \qquad x^2 - 2x + 5 = (x-1)^2 + 4$

$$\int \frac{x+1}{x^2-2x+5}\,dx = \int \frac{(x-1)+2}{(x-1)^2+4}\,dx = \int \frac{y+2}{y^2+4}\,dy$$

$$= \int \frac{y}{y^2+4} + \frac{2}{y^2+4}\,dy = \frac{\log(y^2+4)}{2} + \arctan\left(\frac{y}{2}\right) + C$$

>0

$$= \frac{\log((x-1)^2+4)}{2} + \arctan\left(\frac{x-1}{2}\right) + C$$

WHAT IF THESE THINGS MIX UP?

$$\text{EX: } \int \frac{x^3}{x^3-1}\,dx \underset{\substack{\uparrow \\ (0) \\ \text{LONG DIV}}}{=} \int 1 + \frac{1}{x^3-1}\,dx$$

$$x^3 - 1 \underset{\substack{\uparrow \\ (1)\ \text{FACTOR}}}{=} (x-1)(x^2+x+1) \qquad \frac{1}{x^3-1} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$$

(2) DECOMPOSE

$$\downarrow\; = \frac{-\frac{x}{3} - \frac{2}{3}}{x^2+x+1} + \frac{\frac{1}{3}}{x-1}$$

$$1 = (x-1)(Ax+B) + C(x^2+x+1)$$

$C = \frac{1}{3} \qquad Ax^2 + \frac{x^2}{3} = 0 \quad A = -\frac{1}{3}$

$-B + \frac{1}{3} = 1 \quad B = -\frac{2}{3}$

$$\int 1 + \frac{1}{x^3-1}\,dx \underset{\substack{\uparrow \\ (3) \\ \text{INTEGRATE}}}{=} \frac{1}{3}\int 3 - \frac{x+2}{x^2+x+1} + \frac{1}{x-1}\,dx$$

$$= x + \frac{\log|x-1|}{3} - \frac{1}{3}\int \frac{(x+\frac{1}{2})+\frac{3}{2}}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx =$$

$\leftarrow a^2 = \frac{3}{4},\ a = \frac{\sqrt{3}}{2}$

$$= x + \frac{\log|x-1|}{3} - \frac{\log\left((x+\frac{1}{2})^2+\frac{3}{4}\right)}{6} - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

WE ARE <u>NOT GONNA DO</u> POWERS OF QUAD.

FACTORS.