Integration Techniques: Partial Fractions

1. **Warm-Up**:

   (a) Evaluate $\int \left(\frac{2}{x-3} - \frac{1}{x+5} \right) dx$
   Solution. $2 \log|x-3| + \log|x+5| + C$

   (b) Evaluate $\int \frac{x+13}{x^2+2x-15} dx$

   Solution. Notice that
   $$\frac{x+13}{x^2+2x-15} = \frac{2}{x-3} - \frac{1}{x+5}.$$
   One can verify the equation above by doing cross multiplication.(putting $\frac{2}{x-3} - \frac{1}{x+5}$ into a common denominator) So, the answer is the same as (a).

2. Evaluate $\int \frac{2x-5}{x^2+x-2} dx$

   Solution.

   Step 0: Check degree
   Check if the rational function $f(x) = \frac{P(x)}{Q(x)}$ satisfies degree of $P(x) <$ degree of $Q(x)$. If not, one can apply long division and rewrite the function into the "correct format".

   Step 1: Factor the denominator
   Make sure the denominator is in the form of products of linear and/or irreducible quadratic factors. For this problem, we can factor x^2+x-2 as $(x+2)(x-1)$.

   > *Cool fact*: In principle, every polynomial can be factored as a product powers of distinct linear or irreducible quadratic factors (for example $(x+1)^2(x-3)(x^2+2x+5)$)

   Step 2: Break the integrand $f(x)$ into simpler fractions
   $$\frac{2x-5}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

   Step 3: Find coefficients
   $$\frac{2x-5}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{A(x-1)+B(x+2)}{x^2+x-2}$$
   We want to pick A and B such that the last equality holds for all x. In fact, we just need to make sure the numerators equal to each other, because the denominators are already the same.
   Find constants A and B such that
   $$2x-5 = A(x-1) + B(x+2)$$
   for all x. There are many ways to do this, but my favorite way is the following:

- Plug $x = 1$ into the equation: we get $-3 = A(0) + B(3)$ which implies $B = -1$.
- Plug $x = -2$ into the equation: we get $-9 = A(-3) + B(0)$ which implies $A = 3$.

Hence we can rewrite our function f as the following:

$$\frac{2x-5}{x^2+x-2} = \frac{3}{x+2} - \frac{1}{x-1}$$

(Note: here is another way to find the constants, sometimes you might need use both methods.

$$2x - 5 = A(x-1) + B(x+2) = Ax - A + Bx + 2B = (A+B)x + (-A+2B)$$

we can then compare the coefficient in front of x and the constant term on both sides of the equation, and get a linear system

$$\begin{cases} 2 = A + B \\ -5 = -A + 2B \end{cases}$$

By elimination, we get $A = 3$ and $B = -1$ as well!)

Step 4: Integrate each term
Almost there, we now just need to integrate each term

$$\int \frac{2x-5}{x^2+x-2}\,dx = \int \frac{3}{x+2} - \frac{1}{x-1}\,dx = \boxed{3\log|x+2| - \log|x-1| + C}$$

Don't forget you can always check you get the right answer by taking the derivative!

3. (a) Evaluate $\displaystyle\int \frac{x^3-x-3}{x^2+x-2}\,dx$.
   Solution. Step 0 is not satisfied! So we need to do something before we perform the method of partial fractions. Long division will do the job!

$$\begin{array}{r|l} & x - 1 \\ x^2+x-2 & x^3 \quad\quad - x - 3 \\ & -x^3 - x^2 + 2x \\ \hline & -x^2 \; + x - 3 \\ & x^2 \; + x - 2 \\ \hline & 2x - 5 \end{array}$$

   Hence $\displaystyle\frac{x^3-x-3}{x^2+x-2} = (x-1) + \frac{2x-5}{x^2+x-2}$.
   Notice that the second term is the same as #2 (so we can use the result in #2).

$$\int \frac{x^3-x-3}{x^2+x-2}\,dx = \int (x-1)\,dx + \int \frac{2x-5}{x^2+x-2}\,dx = \boxed{\frac{1}{2}x^2 - x + 3\log|x+2| - \log|x-1| + C}.$$

   (b) Find the form of the partial fraction decomposition for the following integral $\displaystyle\int \frac{x^2-3}{(x+1)^3}\,dx$.

   Solution. Step 0 is satisfied. Step 1 is done! So, let's move on to step 2:

$$\frac{x^2-3}{(x+1)^3} = \frac{A_1}{(x+1)} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3}$$

Then we find the coefficients (step3):

$$\frac{x^2-3}{(x+1)^3} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3} = \frac{A_1(x+1)^2 + A_2(x+1) + A_3}{(x+1)^3}.$$

We want to find A_1, A_2 and A_3 such that $x^2 - 3 = A_1(x+1)^2 + A_2(x+1) + A_3$. Plugging in $x = -1$, we get $-1 = A_3$. It seems like there is no other smart x, but look harder: A_1 is actually very easy to find. Let's compare the coefficient in front of x^2 on both sides of the equation. Notice that the only term involves x^2 on the right hand side is A_1x^2 and on the left hand side is x^2, so we can conclude that $A_1 = 1$. Similarly, we can find A_2 by comparing the coefficient in front of x (it requires a bit more work). The left hand side is 0, and the right hand side is $2A_1 + A_2 = 2 + A_2$. By setting them equal to each other $0 = 2 + A_2$, we get $A_2 = -2$. Our final answer is

$$\int \frac{x^2-3}{(x+1)^3}\,dx = \int \left(\frac{1}{(x+1)} + \frac{-2}{(x+1)^2} + \frac{-1}{(x+1)^3}\right)\,dx$$

You can keep going and evaluate the integral from here (which is step 4.).

(c) Evaluate the integral $\displaystyle\int \frac{x^2-9x+17}{(x-2)^2(x+1)}\,dx$.

Note: we did not cover this question in class, but it's a fun exercise!

Solution. $\displaystyle\frac{x^2-9x+17}{(x-2)^2(x+1)} = \frac{A_1}{x+1} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2} = \frac{A_1(x-2)^2 + A_2(x-2) + A_3(x+1)}{(x-2)^2(x+1)}$

A_1 must be 1, then plug in $x = 2$, we get $3 = 3A_3$, so $A_3 = 1$. For A_2 you can compare the coefficients in front of x (or constant), and it is -6. Hence

$$\int \frac{x^2-9x+17}{(x-2)^2(x+1)}\,dx = \int \left(\frac{1}{x+1} + \frac{-6}{x-2} + \frac{1}{(x-2)^2}\right)\,dx = \boxed{\log|x+1| - 6\log|x-2| - \frac{1}{(x-2)} + C}$$