WARM-UP

- FIND $\int \frac{x+2}{x^2+x+1}$
- REWRITE $\frac{x^2}{(x+1)^2(x+2)^2}$ IN AN EASY TO INTEGRATE FORM

- $x^2+x+1 = (x+a)^2+b = x^2+2ax+a^2+b$

$2a = 1 \sim a = \frac{1}{2}$

$a^2+b = 1 \sim \frac{1}{4}+b = 1 \sim b = \frac{3}{4}$

$x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4} \qquad x+2 = \left(x+\frac{1}{2}\right)+\frac{3}{2}$

$$\int \frac{x+2}{x^2+x+1}\,dx \underset{y=x+\frac{1}{2}}{=} \int \frac{y+\frac{3}{2}}{y^2+\frac{3}{4}}\,dy =$$

$$\int \frac{y}{y^2+\frac{3}{4}}\,dy + \frac{3}{2}\int \frac{1}{y^2+\underbrace{\tfrac{3}{4}}_{b^2}}\,dy =$$

$$\boxed{\int \frac{1}{y^2+b^2}\,dx = \frac{\operatorname{atan}\left(\frac{y}{b}\right)}{b}}$$

$$\frac{\log\left(y^2+\frac{3}{4}\right)}{2} + \frac{3}{2}\,\frac{2}{\sqrt{3}}\operatorname{atan}\left(\frac{2y}{\sqrt{3}}\right) =$$

$$\frac{\log\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)}{2} + \sqrt{3}\operatorname{atan}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\frac{x^2}{(x+1)^2(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

$$0\cdot x^3 + 1\cdot x^2 + 0\cdot x + 0 = A\cdot(x+1)(x+2)^2 + B(x+2)^2 + C(x+2)(x+1)^2$$
$$+ D(x+1)^2$$

<u>$x = -1$</u>

$$1 = A\cdot 0 + B\cdot 1^2 + C\cdot 0 + D\cdot 0 = B \qquad \boxed{B=1}$$

<u>$x = -2$</u>

$$4 = A\cdot 0 + B\cdot 0 + C\cdot 0 + D = D \qquad \boxed{D=4}$$

<u>3rd DEGREE TERM</u>

$$0\cdot x^3 = Ax^3 + Cx^3 \sim A + C = 0 \qquad -A - 4 = 0$$

$$\boxed{A=-4}$$

<u>CONSTANT TERM</u>

$$0 = 4A + 4B + 2C + D = 4A + 4 + 2C + 4$$

$$\sim C = -2A - 4$$

$$\boxed{C = 4}$$

$$\frac{x^2}{(x+1)^2(x+2)^2} = \frac{-4}{x+1} + \frac{1}{(x+1)^2} + \frac{4}{x+2} + \frac{4}{(x+2)^2}$$

NUMERICAL INTEGRATION

AS WE'VE SEEN, EVALUATING AN INTEGRAL CAN BE HARD, AND IT MIGHT BE THAT WE CANNOT DO IT AT ALL FOR SOME $f(x)$. SAME GOES FOR FINDING THE EXACT LIMIT OF A SEQUENCE OF RIEMANN SUMS.

SO WHAT CAN WE DO? OR MORE PRECISELY, WHAT CAN WE DO THAT WE CAN ALSO PROGRAM A COMPUTER TO DO?

IDEA: WE COULD PICK A RIEMANN SUM WITH n RECTANGLES, AND IF n IS BIG ENOUGH IT WILL BE VERY CLOSE TO THE ACTUAL INTEGRAL.

MORE GENERAL IDEA: WE CAN JUST DIVIDE OUR INTEGRAL IN n EQUAL PARTS AND IN EACH OF THESE USE SOME REASONABLE APPROXIMATION OF $f(x)$ THAT WE CAN INTEGRATE; THEN ADD UP THE PIECES.

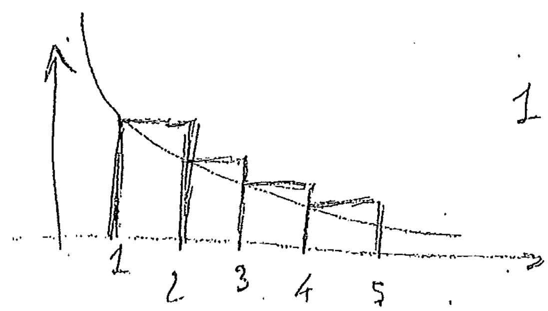
REMARK 1 PICKING A RIEMANN SUM IS JUST APPROXIMATING THE FUNCTION WITH A CONSTANT

REMARK 2 AS IT HAPPENS IN GENERAL WITH APPROXIMATIONS (REMEMBER LINEAR AND QUADRATIC APPROX) WE WILL NEED AN ERROR ESTIMATE, OTHERWISE OUR APPROXIMATIONS WILL BE USELESS.

EXAMPLE:

WE WANT TO APPROXIMATE $\int_1^5 \frac{1}{x}\,dx = \log(5) \approx 1.60943$ IN $n=4$ STEPS (INTERVALS)

LEFTPOINT RULE

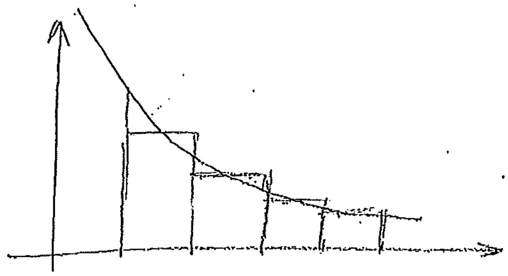


$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}$

≈ 2.08333

$|\text{ERROR}| \approx 0.47$

MIDPOINT RULE

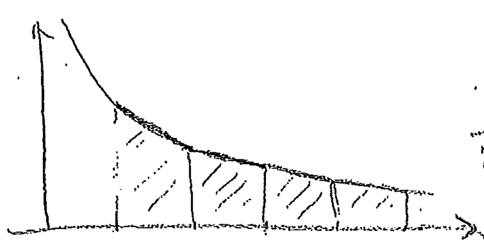


$\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\frac{2}{9}=\frac{496}{315}$

≈ 1.57460

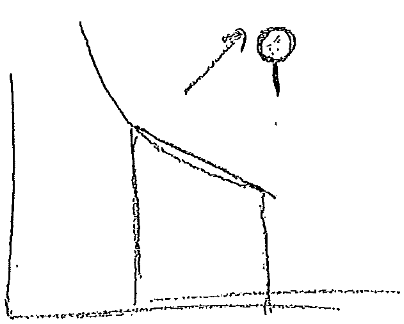
$|\text{ERROR}| \approx 0.034$

TRAPEZOID RULE



$\frac{1}{2}\left(\left(1+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}\right)\right)$

$=\frac{101}{60} \approx 1.68333$



$|\text{ERROR}| \approx 0.073$

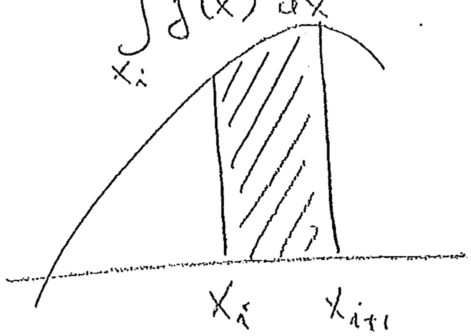
WE'LL STUDY THREE TECHNIQUES OF NUMERICAL INTEGRATION:

MIDPOINT RULE, TRAPEZOIDAL RULE AND SIMPSON'S RULE (UNRELATETAD TO THE SIMPSONS, UNFORTUNATELY)

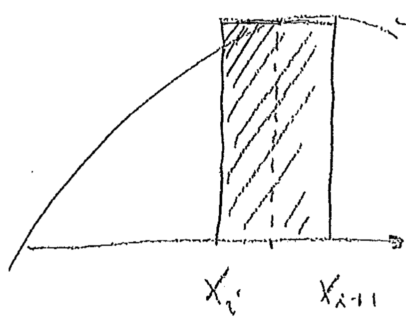
MIDPOINT RULE

WE APPROXIMATE THE INTEGRAL BY JUST TAKING A MIDPOINT RIEMANN SUM. THIS IS SURPRISINGLY EFFECTIVE!

$$\int_{x_i}^{x_{i+1}} f(x)\,dx$$



$\sim$



$$\int_{x_i}^{x_{i+1}} f\left(\frac{x_i+x_{i+1}}{2}\right)dx$$

$$\parallel$$

$$f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$$

SO WHAT WE ARE DOING IS APPROXIMATING

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \quad \text{WITH} \quad f\left(\frac{x_i+x_{i+1}}{2}\right)\cdot(x_{i+1}-x_i)$$

DEF: LET $\Delta_x = \frac{b-a}{n}$. THE MIDPOINT RULE APPROX (WITH n STEPS) IS:

$$\int_a^b f(x)\,dx \simeq \left[f\left(a+\frac{\Delta_x}{2}\right)+f\left(a+\frac{3}{2}\Delta_x\right)+\dots+f\left(a+\left(n-\frac{1}{2}\right)\Delta_x\right)\right]\Delta_x$$

EX: $\int_0^1 \frac{4x}{1+x^2}\,dx$, APPROX WITH MID RULE, $M=4$

STEPS

- THE X VALUES: $a=0$, $b=1$, $\Delta x = \frac{1}{4}$

SO THE X VALUES WILL BE

$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$

- $$\int_0^1 \frac{4x}{1+x^2}\,dx \approx \left[\frac{\frac{1}{2}}{1+\frac{1}{64}} + \frac{\frac{3}{2}}{1+\frac{9}{64}} + \frac{\frac{5}{2}}{1+\frac{25}{64}} + \frac{\frac{7}{2}}{1+\frac{49}{64}} \right] \cdot \frac{1}{4}$$

$= 1.396857\ldots$

LET'S SEE HOW GOOD OUR APPROX IS;

$$\int_0^1 \frac{4x}{1+x^2}\,dx = \log(4) = 1.386294\ldots$$

SO THE ERROR IS $|\log(4) - 1.396857\ldots| = 0.01056\ldots$

$\approx 10^{-2}$, OR $\approx 1\%$ OF THE ORIGINAL QTY.

DEF: SUPPOSE α IS AN APPROX OF A.

- ABSOLUTE ERROR $= |A - \alpha|$
- RELATIVE ERROR $= \frac{|A-\alpha|}{A}$
- % ERROR $= \frac{|A-\alpha|}{A} \cdot 100$