## WARM-UP

FIND $\int e^{ax}\sin(bx)\,dx$.

SOL: IBP, $g = e^{ax}$, $F = \sin(bx)$

$$* \quad \frac{e^{ax}}{a}\cdot\sin bx - \int \frac{e^{ax}}{a}\cdot b\cos(bx)\,dx = \int e^{ax}\sin(bx)\,dx$$

IBP ON $\int \frac{e^{ax}}{a} b\cos(bx)\,dx$

$$** \quad \frac{b}{a^2}\cos bx + \frac{b^2}{a^2}\int e^{ax}\sin(bx)\,dx = \frac{a}{b}\int e^{ax}\cos(bx)\,dx$$

PLUG INTO $*$

$$e^{ax}\left(\frac{\sin bx}{a} - \frac{b\cos bx}{a^2}\right) + C = \left(1+\frac{b^2}{a^2}\right)\int e^{ax}\sin bx\,dx$$

VERIFY: $\frac{d}{dx}\, e^{ax}\left(\frac{\sin bx}{a} - \frac{b\cos bx}{a^2}\right) =$

$$a e^{ax}\left(\frac{\sin bx}{a} - \frac{b\cos bx}{a^2}\right) + e^{ax}\left(\frac{b}{a}\cos bx + \frac{b^2}{a^2}\sin bx\right)$$

$$= e^{ax}\left(\sin bx - \frac{b}{a}\cos bx + \frac{b}{a}\cos bx + \frac{b^2}{a^2}\sin bx\right)$$

$$= \left(1+\frac{b^2}{a^2}\right) e^{ax}\sin bx. \quad \checkmark$$

# TRIGONOMETRIC INTEGRALS

TRIG FUNCTIONS ARE POSSIBLY THE MOST UBIQUITOUS FUNCTIONS IN PHYSICS AND ENGINEERING. IT'S ONLY NATURAL THAT WE'D WANT TO HAVE AN ALGORITHM TO INTEGRATE SIMPLE COMBINATIONS OF THESE FUNCTIONS.

FIRST WE HAVE TO RECALL SOME IMPORTANT TRIGONOMETRIC IDENTITIES:

- $\sin(x)^2 + \cos(x)^2 = 1$
- $\sin(2x) = 2\sin x \cos x$
- $\cos(2x) = \cos(x)^2 - \sin(x)^2 = 2\cos(x)^2 - 1 = 1 - 2\sin(x)^2$
- $\sin(x)^2 = \dfrac{1-\cos(2x)}{2}$
- $\cos(x)^2 = \dfrac{1+\cos(2x)}{2}$

THE LAST TWO ALLOW US TO REWRITE HIGHER POWERS OF $\sin(x)$, $\cos(x)$ IN TERMS OF LOWER POWERS

EXAMPLE: $\cos(x)^4 = \left(\dfrac{1+\cos(2x)}{2}\right)^2 = \dfrac{1}{4}\left(\underbrace{\cos(2x)^2} + 2\cos(2x) + 1\right)$

$= \dfrac{1}{4}\left(\underbrace{\dfrac{1+\cos(4x)}{2}} + 2\cos 2x + 1\right) = \dfrac{\cos(4x)}{8} + \dfrac{\cos(2x)}{2} + \dfrac{3}{8}$

SO WHILE WE WOULD HAVE TO COME UP WITH SOME IDEA TO COMPUTE $\int \cos(x)^4$ DIRECTLY, IT IS IMMEDIATE USING THE IDENTITY ABOVE.

WE'LL USE SIMILAR TRICKS, IN ADDITION TO THE INTEGRATION TECHNIQUES WE DEVELOPED TO SOLVE TWO FAMILY OF INTEGRALS, THA IS

$$\int \sin(x)^m \cos(x)^n \, dx \quad \text{AND}$$

$$\int \tan(x)^m \sec(x)^n \, dx$$

FOR INTEGERS $n, m$. THE DETAILS WILL DEPEND ON $n, m$ BEING ODD OR EVEN.

## THE INTEGRAL OF $\sin(x)^m \cos(x)^n$

CASE 1: ONE OF THEM IS ODD $> 0$

( ~~EXACTLY ONE~~ OR BOTH )

EXAMPLE: $\int \overset{u}{\sin(x)^2} \, \overset{u'}{\cos(x)} \, dx = \int u^2 u' \, dx \Big|_{u=\sin(x)}$

$$= \int u^2 \, du \Big|_{u=\sin x} = \frac{u^3}{3} + C \Big|_{u=\sin(x)} = \frac{\sin(x)^3}{3} + C$$

IDEA:

- PICK THE ODD ONE (IF BOTH, PICK EITHER); SAY WE PICK $\cos x$.
- WRITE $\cos x^n = \cos x^{2a+1}$; SO $\sin x^m \cos x^n$ IS $\sin x^m \cos x^{2a+1} = \sin x^m (1-\sin^2 x)^a \cdot \cos x$.
- SUBSTITUTE $u = \sin x$, $u' = \cos x$, AND INTEGRATE $u^m (1-u^2)^a$
- REMEMBER TO PLUG $u = \sin x$ BACK IN!

Ex: $\int \sin(x)^3 \cos(x)^2 \, dx = \int \cos(x)^2 (1-\cos(x)^2) \sin x \, dx$

$\boxed{u = \cos x, \quad u' = -\sin x}$ $= -\int u^2 - u^2 \, du \Big|_{u=\cos x}$

$= \frac{u^5}{5} - \frac{u^3}{3} \Big|_{u=\cos x} = \frac{\cos(x)^5}{5} - \frac{\cos(x)^3}{3} + C$

VERIFY: $\frac{d}{dx}\left( \frac{\cos(x)^5}{5} - \frac{\cos(x)^3}{3} \right) =$

$\frac{-5 \sin x \cos(x)^4}{5} + \frac{3 \sin x \cos(x)^2}{3} =$

$= \sin x (\cos(x)^4 - \cos(x)^2) = \sin x (\cos x)^2 (1 - \cos(x)^2)$

$= \sin^3 x \cos^2 x$ ✓

BOTH $m, n$ EVEN $\geq 0$

IDEA: USE TRIG IDENTITIES TO GET BACK TO PREV CASE

EX: $$\int \cos(x)^2 dx = \int \frac{1+\cos(2x)}{2} dx = \frac{2x+\sin(2x)}{4} + C$$

EX: $$\int \cos(x)^2 \sin(x)^2 dx = \int \frac{1+\cos(2x)}{2} \cdot \frac{1-\cos(2x)}{2} dx$$

$$= \frac{1}{4} \int 1-\cos(2x)^2 dx = \frac{1}{4} \int 1 - \frac{1+\cos(4x)}{2} dx$$

$$= \int \frac{1}{8} dx - \int \frac{\cos(4x)}{8} dx = \frac{x}{8} - \frac{\sin(4x)}{32} + C$$

VERIFY (FIRST ONE): $\frac{d}{dx} \frac{2x+\sin(2x)}{4} =$

$$\frac{1}{2} + \frac{2\cos 2x}{4} = \frac{1+\cos 2x}{2} = \cos(x)^2 \checkmark$$

MORE PRECISELY:

- USE $\sin^2 x = \frac{1-\cos(2x)}{2}$, $\cos^2 x = \frac{1+\cos(2x)}{2}$ TO DIVIDE DEGREE BY 2, EXPAND EXPRESSION
- WHEN A COMPONENT HAS ODD FACTOR, GO BACK TO PREVIOUS ALGORITHM. WHEN STILL BOTH EVEN, APPLY FIRST POINT AGAIN.

# THE INTEGRAL OF $\tan(x)^m \sec(x)^n$

RECALL THAT

$$\sec(x) = \frac{1}{\cos(x)}, \quad (\tan(x))' = \sec(x)^2,$$

$$\sec(x)' = \sec(x)\tan(x) \qquad 1 + \tan(x)^2 = \sec(x)^2$$

THERE ARE FIVE BASIC CASES:

① $m$ ODD, $n$ ANY (EVEN FRACTIONAL!)

$$(\tan(x))^m (\sec(x))^n = \frac{\sin(x)^m}{\cos(x)^{m+n}} = \frac{\sin(x)^{m-1}}{\cos(x)^{m+n}} \sin(x)$$

$u = \cos(x)$

② ALTERNATIVELY $m$ ODD, $n \geq 1$

$$\tan(x)^m \sec(x)^n = \tan(x)^{m-1} \sec(x)^{n-1} \cdot \sec(x)\tan(x)$$

$u = \sec(x) \quad u' = \tan(x)\sec(x), \quad \tan(x)^{m-1} =$
$(\sec(x)^2 - 1)^{\frac{m-1}{2}}$

③ $n$ EVEN $\geq 2$ $\quad \tan(x)^m \sec(x)^n =$

$$\tan(x)^m \sec(x)^{n-2} \sec(x)^2 \qquad u = \tan(x)$$
$$u' = \sec(x)^2$$

THEN USE $\sec(x)^2 = \tan(x)^2 + 1$

④ $m$ EVEN, $n = 0$ $\quad \tan(x)^m = \tan(x)^{m-2}(\sec(x)^2 - 1)$
REPEATEDLY AND THEN $u = \tan(x)$

(5) m ODD , m EVEN.

TOO COMPLICATED FOR NOW, WE'LL GET BACK TO IT LATER IN THE COURSE.

AN EXAMPLE FOR EACH CASE:

EXAMPLE:

$$\int \operatorname{Tan}(x)^3 \sec(x)^{\frac{1}{2}}\, dx = \int \frac{\sin(x)^3}{\cos(x)^{\frac{7}{2}}}\, dx$$

$u = \cos(x)$

$$\int \frac{\sin(x)^3}{\cos(x)^{\frac{7}{2}}}\, dx = \int \frac{-(1-u^2)}{u^{\frac{7}{2}}} \cdot u'\, dx \Big|_{u=\cos(x)}$$

$$= -\int u^{-\frac{7}{2}} - u^{-\frac{3}{2}}\, du \Big|_{u=\cos(x)} = -\frac{2}{5} u^{-\frac{5}{2}} + 2 u^{-\frac{1}{2}} + C \Big|_{u=\cos x}$$

$$= -\frac{2}{5}\cos(x)^{-\frac{5}{2}} + 2\cos(x)^{-\frac{1}{2}} + C$$ (VERIFY!)

EXAMPLE:

$$\int \operatorname{Tan}(x)^3 \sec(x)^4\, dx = \int \operatorname{Tan}(x)^2 \sec(x)^2 \cdot \operatorname{Tan}(x) \sec(x)^2\, dx$$

$\boxed{u = \sec(x)}$

$$= \int (u^2-1)^2 u^3 \cdot u'\, dx = \int (u^2-1)^2 u^3\, du \Big|_{u=\sec x}$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C \Big|_{u=\sec(x)} = \frac{\sec(x)^6}{6} - \frac{\sec(x)^4}{4} + C$$ (VERIFY!)