

WE WILL USE THE FOLLOWING
THREE SUBSTITUTIONS:

$$\sqrt{a^2 - x^2} \quad : \quad x = a \sin(u)$$

$$\sqrt{a^2 + x^2} \quad : \quad x = a \tan(u)$$

$$\sqrt{x^2 - a^2} \quad : \quad x = a \sec(u)$$

SO THAT

$$\sqrt{a^2 - x^2} \rightsquigarrow \sqrt{a^2 - a^2 \sin^2(u)} = \sqrt{a^2 \cos^2(u)} = |a \cos(u)|$$

$$\sqrt{a^2 + x^2} \rightsquigarrow \sqrt{a^2 + a^2 \tan^2(u)} = \sqrt{a^2 \sec^2(u)} = |a \sec(u)|$$

$$\sqrt{x^2 - a^2} \rightsquigarrow \sqrt{a^2 \sec^2(u) - a^2} = \sqrt{a^2 \tan^2(u)} = |a \tan(u)|$$

EXAMPLE :

$$\int \frac{x^2}{\sqrt{1-x^2}} dx \quad \begin{array}{l} \text{sim}(u) = x \\ \downarrow \\ \int \frac{\text{sim}(u)^2}{\sqrt{1-\text{sim}(u)^2}} \end{array} \quad \begin{array}{l} \frac{dx}{du} = \cos(u) \\ \downarrow \\ \cos(u) \end{array} du =$$

$$\int \frac{\text{sim}(u)^2 \cos(u)}{\sqrt{\cos^2(u)}} du = \int \text{sim}(u)^2 du =$$

$$\begin{array}{l} \uparrow \\ \text{POSITIVE ON } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array} \quad \frac{1}{2} \int (1 - \cos(2u)) du =$$

$$= \frac{u}{2} - \frac{1}{4} \sin(2u) + C \quad ; \quad \text{NOW} \quad \begin{array}{l} x = \text{sim}(u) \\ u = \text{aresim}(x) \end{array}$$

$$= \frac{\text{aresim}(x)}{2} - \frac{1}{4} \sin(2 \text{aresim}(x)) + C$$

IT'S NOT OVER !

$$\frac{\arcsin(x)}{2} - \frac{1}{4} \sin(2 \arcsin(x)) + C =$$

$$\frac{\arcsin(x)}{2} - \frac{1}{4} \cdot 2 \sin(\arcsin(x)) \cos(\arcsin(x)) + C$$

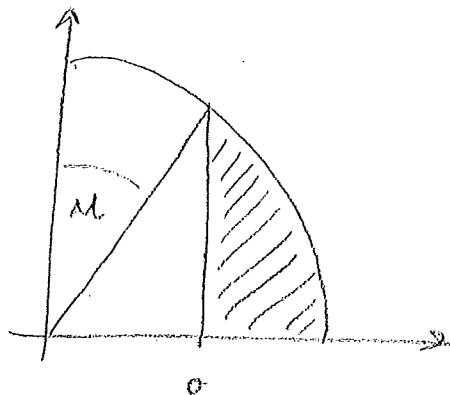
$$= \frac{\arcsin(x)}{2} - \frac{1}{2} \cdot x \cdot \sqrt{1-x^2} + C \quad (\text{VERIFY!})$$

EXAMPLE:

$$\int_a^r \sqrt{r^2 - x^2} dx$$

$$x = r \sin(u)$$

$$\text{AND } u = \arcsin \frac{x}{r}$$



EXTREMES OF INTEGRATION:

$$u = \arcsin \frac{a}{r}$$

$$u = \arcsin \frac{r}{r} = \arcsin(1) = \frac{\pi}{2}$$

$$\text{So } \int_a^r \sqrt{r^2 - x^2} dx = \int_{\arcsin \frac{a}{r}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2(u)} \cdot r \cos(u) du$$

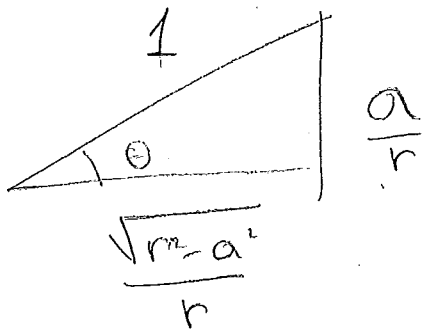
$$= \int_{\arcsin \frac{a}{r}}^{\frac{\pi}{2}} r^2 \sqrt{1 - \sin^2(u)} \cdot \cos(u) du = r^2 \int_{\arcsin \frac{a}{r}}^{\frac{\pi}{2}} \cos^2(u) du$$

$$= r^2 \int \frac{1 + \cos(2u)}{2} du = \frac{r^2}{2} \left(u + \frac{\sin(2u)}{2} \right) \Big|_{\arcsin \frac{a}{r}}^{\frac{\pi}{2}}$$

$$= \frac{r^2}{2} \left(\frac{\pi}{2} + \frac{\sin(\pi)}{2} - \arcsin\left(\frac{a}{r}\right) - \frac{\sin\left(2 \arcsin\left(\frac{a}{r}\right)\right)}{2} \right)$$

$$= \frac{r^2}{2} \left(\frac{\pi}{2} - \arcsin\left(\frac{a}{r}\right) - \frac{2 \sin\left(\arcsin\left(\frac{a}{r}\right)\right) \cos\left(\arcsin\left(\frac{a}{r}\right)\right)}{2} \right)$$

$$= \frac{r^2}{2} \left(\frac{\pi}{2} - \arcsin\left(\frac{a}{r}\right) - \frac{a}{r} \sqrt{1 - \frac{a^2}{r^2}} \right)$$



EXAMPLE: $\int \frac{1}{x^2 \sqrt{9+x^2}} dx$

$\sqrt{9+x^2} = \sqrt{3^2+x^2}$ so we pick

$x = 3 \tan u$ ($u = \arctan\left(\frac{x}{3}\right)$)

so $\sqrt{9+x^2} = \sqrt{9+9 \tan^2(u)} = 3 \sqrt{\sec^2(u)}$

$$\int \frac{1}{x^2 \sqrt{9+x^2}} dx = \int \frac{1}{27 \tan^2(u) |\sec(u)|} \cdot 3 \sec(u)^2 du \Big|_{u=u(x)}$$

$$= \frac{1}{9} \int \frac{\sec(u)}{\tan^2(u)} du \Big|_{u=u(x)} = \frac{1}{9} \int \frac{\cos(u)}{\sin^2(u)} du \Big|_{u=u(x)}$$

NOW WE CAN WRITE $y = \sin u$

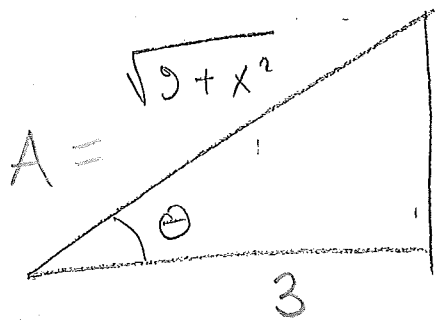
$$\frac{1}{9} \int \frac{\cos(u)}{\sin(u)^2} du = \frac{1}{9} \int \frac{y'}{y^2} du = \frac{1}{9} \int y^{-2} dy \Big|_{y=y(u)}$$

$$= -\frac{1}{9} y^{-1} + C \Big|_{y=y(u)} = -\frac{1}{9} \sin(u) + C$$

NOW WE HAVE TO RECALL THAT
 $u = \arctan\left(\frac{x}{3}\right)$

$$\int \frac{1}{x^2 \sqrt{9+x^2}} dx = -\frac{1}{9} \frac{1}{\sin(\arctan(\frac{x}{3}))} + C$$

WHAT'S $\sin(\arctan(\frac{x}{3}))$?



$$\sin(\theta) = \frac{B}{A}$$
~~$$= \frac{x}{\sqrt{9+x^2}}$$~~

$$B = A \sin \theta = \frac{x}{\sqrt{x^2+9}}$$

$$\int \frac{1}{x^2 \sqrt{9+x^2}} dx = -\frac{1}{9} \frac{\sqrt{x^2+9}}{x} + C$$

VERIFY: $\frac{d}{dx} \left(-\frac{\sqrt{x^2+9}}{9x} \right) = \frac{2x^2}{2\sqrt{x^2+9}} - \frac{\sqrt{x^2+9}}{9x^2} = -\frac{-18}{18x^2 \sqrt{x^2+9}} = \frac{1}{x^2 \sqrt{x^2+9}} \checkmark$

EXAMPLE: $\int \frac{x^3}{\sqrt{x^2-1}} dx$

OUR LIST OF SUBSTITUTIONS SAYS WE NEED

$x = \sec(u)$, so

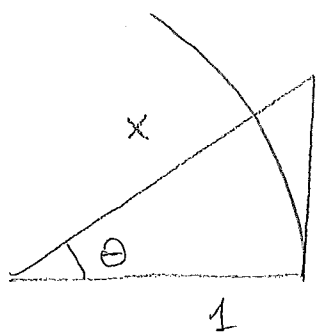
$$\int \frac{x^3}{\sqrt{x^2-1}} dx = \int \frac{\sec(u)^3}{\sqrt{\tan(u)^2}} \cdot \sec(u) \cdot \tan(u) du$$

$$= \int \sec(u)^4 du \stackrel{y = \tan(u)}{\downarrow} \int (y^2+1) y' dy = \int (y^2+1) dy \Big|_{y=y(u)}$$

$$= \frac{y^3}{3} + y + C \Big|_{y=\tan(u)} = \frac{\tan(u)^3}{3} + \tan(u) + C$$

Now $u = \operatorname{arcsec}(x)$

||



$$\frac{(\sqrt{x^2-1})^3}{3} + \sqrt{x^2-1} + C$$

EXAMPLE: $\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1} dx$

WE FIRST REWRITE THE 2nd-DEGREE POLYNOMIAL IN ONE OF THE STANDARD FORMS

$$x^2 - 2x - 3 = (x-a)^2 + b = x^2 - 2ax + a^2 + b$$

So $a=1$, $b = -3 - a^2 = -4$

$$\int_3^5 \frac{\sqrt{x^2 - 2x - 3}}{x-1} dx = \int_3^5 \frac{\sqrt{(x-1)^2 - 4}}{x-1} dx$$

$$= \int_2^4 \frac{\sqrt{y^2 - 4}}{y} dy = \int_{0 = \operatorname{arcsec}(1)}^{\pi/3 = \operatorname{arcsec}(2)} \frac{2\sqrt{\sec(u)^2 - 1} \cdot \sec(u) \tan(u) du}{\sec(u)}$$

$$y = 2 \sec(u)$$

$$u = \operatorname{arcsec}\left(\frac{y}{2}\right)$$

$$= \int_0^{\pi/3} 2 \tan(u)^2 du = 2 \int_0^{\pi/3} \sec(u)^2 - 1 du$$

$$= 2 \tan(u) - u + C \Big|_0^{\pi/3} = 2\sqrt{3} - \frac{2}{3}\pi$$