EX: $\int x^2 e^x \, dx$

SAME IDEA: $F(x) = x^2$, $g(x) = e^x$

$$\underbrace{x^2 e^x}_{FG} - 2\underbrace{\int x e^x dx}_{\int g G} = \int x^2 e^x \, dx$$

$$x^2 e^x - 2x e^x + 2e^x + C = \int x^2 e^x \, dx \quad \text{(VERIFY!)}$$

WE CAN INDUCTIVELY COMPUTE $\int x^m e^x \, dx$

FOR ALL m!

EX: $\int x \log x \, dx$

$$x \begin{array}{l} \nearrow \frac{x^2}{2} \\ \searrow 1 \end{array} \qquad \log x \begin{array}{l} \nearrow ? \\ \searrow \frac{1}{x} \end{array}$$

WE CAN ONLY $\frac{d}{dx}$

SO $F(x) = \log x$, $g(x) = x$

$$\frac{x^2 \log x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \int x \log x \, dx$$

$$\frac{x^2 \log x}{2} - \frac{x^2}{4} + C = \int x \log x \, dx \quad \text{(VERIFY!)}$$

WE CAN DO THIS FOR ALL m!

EX: $\int x^m \log x \, dx \quad m \neq -1$

$$\log x = F(x), \quad x^m = g(x)$$

$$\frac{x^{m+1}}{m+1}\log x - \int \frac{x^{m+1}}{m+1} \cdot \frac{1}{x} \, dx = \int x^m \log x \, dx$$

$$\frac{x^{m+1}}{m+1}\log x - \frac{x^{m+1}}{(m+1)^2} + C = \int x^m \log x \, dx$$

(VERIFY!)

WAIT, SO FOR $m = 0$...

EX: $\int \log x \, dx$

$$\log x = \underset{g(x)}{1} \cdot \underset{F(x)}{\log x}$$

$$x \log x - \int x \cdot \frac{1}{x} \, dx = \int \log x \, dx$$

$$x \log x - x = \int \log x \, dx$$

VERIFY: $\frac{d}{dx}(x \log x - x) = \log x - \frac{x}{x} - 1 = \log x.$

NOTE: THIS IS A COMMON AND IMPORTANT TRICK!

EX: $\int \arctan x \, dx$

NO IDEA... LET'S TRY THE TRICK

$$\arctan x = \underset{\substack{\uparrow \\ g(x)}}{1} \cdot \underset{\substack{\uparrow \\ F(x)}}{\arctan x}$$

$$\underbrace{x \arctan x}_{GF} - \underbrace{\int \frac{x}{1+x^2} dx}_{\int Gf} = \int \arctan x \, dx \qquad u = 1+x^2$$

$$x \arctan x - \int \frac{1}{2} \frac{u'}{u} dx = \int \arctan x \, dx$$

$$x \arctan x - \frac{\log|1+x^2|}{2} + C = \int \arctan x \, dx$$

$$\overset{\shortparallel}{\frac{\log(1+x^2)}{2}} \text{ AS } 1+x^2 > 0$$

EX: $\int (\log x)^2 dx$

$\log x \nearrow x\log - x$

$\log x \searrow \frac{1}{x}$

$$\underbrace{\log x\,(x\log x - x)}_{FG} - \int \underbrace{\log x - 1\, dx}_{fG} = \int \log^2(x)\,dx$$

$$x\log^2 x - 2x\log x + 2x = \int \log^2(x)\,dx$$

VERIFY: $\frac{d}{dx}\left(x\log^2 x - 2x\log x + 2x\right) =$

$$\log^2 x + \underline{2\log x} - 2\log x - 2 + 2 = \log^2 x \quad \checkmark$$

A SUBTLE EXAMPLE

Ex: $\int e^x \sin x\, dx$

$F = \sin x \quad , \quad g = e^x$

$$e^x \sin x - \int e^x \cos x\, dx = \int e^x \sin x\, dx$$

NOT MUCH BETTER... BUT, LET'S TRY AGAIN ON $\int e^x \cos x\, dx$

$$e^x \sin x - e^x \cos x + \int e^x(-\sin x)\,dx = \int e^x \sin\, dx$$

$$e^x \sin x - e^x \cos x + C = 2\int e^x \sin x\, dx$$

SO $\int e^x \sin x = \dfrac{e^x \sin x - e^x \cos x}{2} + C$!

VERIFY:

$$\frac{d}{dx}\left(\frac{e^x \sin x - e^x \cos x}{2}\right) = \frac{e^x \sin x + e^x \cos x}{2}$$

$$- \frac{e^x \cos x - e^x \sin x}{2} = \frac{2 e^x \sin x}{2} \quad \checkmark$$

EX: $\displaystyle\int_0^{2\pi} x^2 \cos x \, dx$

x^2: $\frac{x^3}{3}$, $2x$

$\cos x$: $\sin x$, $-\sin x$

WE INTEGRATE $\cos x$, DIFFERENTIATE x^2

$$\underbrace{x^2 \sin x \Big|_0^{2\pi}}_{=\,0} - \int_0^{2\pi} 2x \sin x \, dx = \int_0^{2\pi} x^2 \cos x \, dx$$

AGAIN, INTEGRATE $\sin$, DIFF $2X$

$$\underbrace{-2x \cos x \Big|_0^{2\pi}}_{=\,-4\pi} + \int_0^{2\pi} 2 \cos x \, dx = \int_0^{2\pi} 2x \sin x \, dx$$

$$-4\pi \quad - \underbrace{2 \sin x \Big|_0^{2\pi}}_{=\,0} = \int_0^{2\pi} 2x \sin x \, dx$$

SO

$$4\pi = -(-4\pi) = \int_0^{2\pi} x^2 \cos x \, dx$$