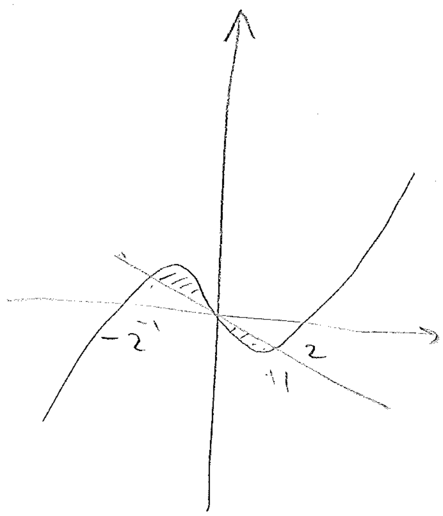
WARM-UP: FIND AREA BETWEEN

$y = x^3 - 4x$ AND $y = -3x$

INTERS: $-3x = x^3 - 4x \sim x^3 - x = 0$ $\boxed{\pm 1, 0}$



$$\int_{-1}^{0} x^3 - x \, dx +$$

$$\int_{0}^{1} x - x^3 \, dx$$

$$= 2 \int_{0}^{1} x - x^3 \, dx =$$

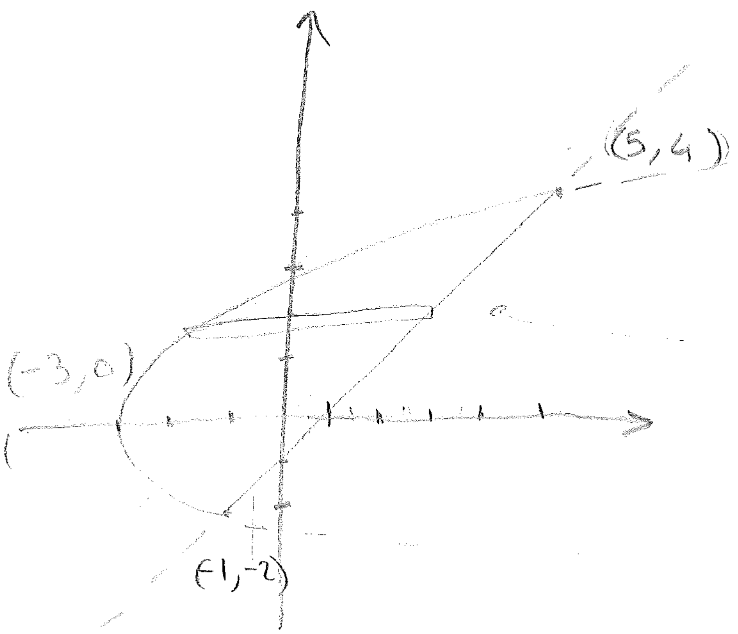
$$2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

A MORE INTRICATE EXAMPLE:

CURVES DO NOT NEED TO BE IN THE FORM $y = f(x)$!

FIND AREA ENCLOSED BY

$\underbrace{y^2 = 2x + 6}_{\text{PARABOLA FACING RIGHT}}$ AND $y = x - 1$



APPROACH 1 (SMART):

SEE AS FUNCTIONS OF y

$x = \frac{y^2}{2} - 3$ $\quad f(y)$

$x = y + 1$ $\quad g(y)$

$f(y) - g(y) = \frac{y^2}{2} - y - 4$ $\quad y^2 - 2y - 8 = 0$

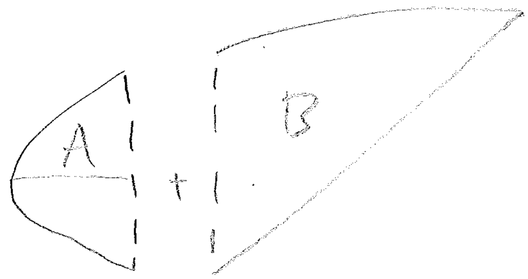
IF $y = -2, 4$ $\quad f(1) = -\frac{5}{2}$, $g(1) = 2$ SO $g \geq f$

$$\int_{-2}^{4} g(y) - f(y)\,dy = \int_{-2}^{4} -\frac{y^2}{2} + y + 4\,dy = -\frac{y^3}{6} + \frac{y^2}{2} + 4y \Big|_{-2}^{4}$$

$$= \left(-\frac{64}{6}+\frac{16}{2}+16\right)-\left(-\frac{8}{6}+\frac{4}{2}-8\right)=18$$

APPROACH 2 (NOT AS SMART)

SPLIT AS



THEN USE $y=\pm\sqrt{2x+6}$

A: $f(x)=\sqrt{2x+6}$ $\quad g(x)=-\sqrt{2x+6}$

B: $f(x)=\sqrt{2x+6}$ $\quad g(x)=x-1$

$$A: \int_{-3}^{-1} 2\sqrt{2x+6}\,dx \underset{u=2x+6}{=} \int_0^4 \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_0^4 = \frac{16}{3}$$

$$B: \int_{-1}^{5}\sqrt{2x+6}-x+1\,dx = \frac{1}{2}\int_4^{16}\sqrt{u}\,du - \int_{-1}^{5} x-1\,dx$$

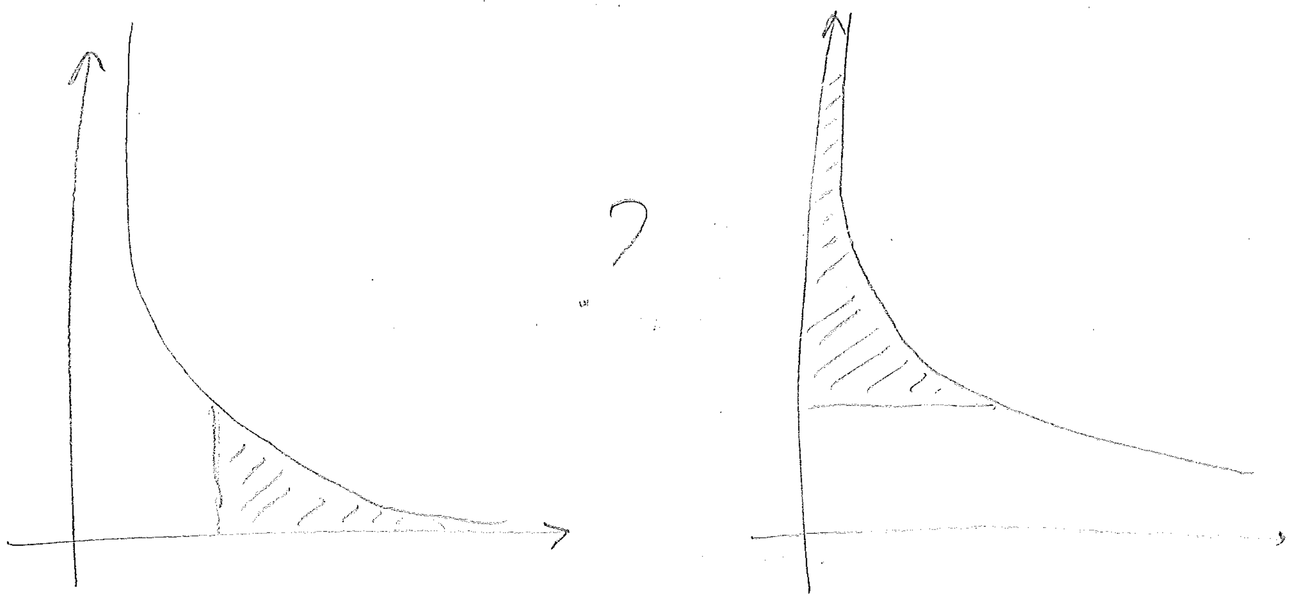
$$=\left[\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\right]_4^{16} - \left[\frac{x^2}{2}-x\right]_{-1}^{5} = \frac{64}{3}-\frac{8}{3}-\frac{15}{2}+\frac{3}{2}=$$

$$-6+\frac{56}{3}$$

$$A+B=\frac{56}{3}+\frac{16}{3}-6=\frac{72}{3}-6=24-6=18$$

EX:

DO THE "AREA UNDER $y = \frac{1}{x^3}$ AND TO THE RIGHT OF $x = 1$" AND THE "AREA UNDER $y = \frac{1}{x^{\frac{1}{3}}}$ AND ABOVE $y = 1$" MAKE SENSE?



NOTE: $y = \frac{1}{x^3} \sim x = \frac{1}{y^{\frac{1}{3}}}$ SO THE TWO REGIONS ARE "THE SAME"

HOW TO COMPUTE? TAKE A LIMIT

$$\lim_{R\to\infty} \int_1^R \frac{1}{x^3}\,dx = \lim_{R\to\infty} \left[-\frac{1}{2x^2}\right]_1^R = \lim_{R\to\infty} -\frac{1}{2R^2} + \frac{1}{2}$$

$$= \frac{1}{2} - \lim_{R\to\infty} \frac{1}{2R^2} = \frac{1}{2} \;!$$

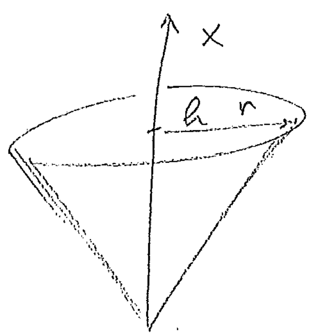
VOLUMES:

WE MOVE ON TO THE REALM OF THREE DIMENSIONAL SHAPES.

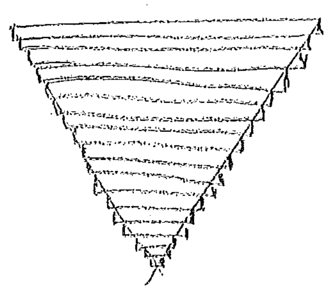
MOST THREE DIMENSIONA SHAPES REQUIRE MULTI-VARIABLE INTEGRALS TO COMPUTE, BUT IN SOME CASES WE CAN USE THE SIMMETRIES OF THE PROBLEM TO REDUCE TO ONE VARIABLE.

EXAMPLE:

CONE OF HEIGHT h, RADIUS r



WE SLICE (APPROXIMATE) THE CONE INTO "PANCAKES"

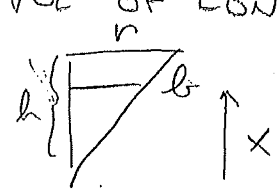


AS THE NUMBER OF PANCAKES GOES TO ∞

VOL OF PANCAKE $= \Delta_x$ AREA

$$\sum \text{VOL OF PANC.} = \int \text{AREA}\, dx = \text{VOL OF CONE}$$

AREA OF SLICE AT HEIGHT x =

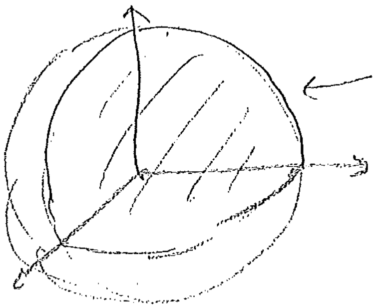


$$\frac{r}{h} = \frac{b}{x} \sim b = \frac{xr}{h} \qquad \text{AREA} = \pi \frac{x^2 r^2}{h^2}$$

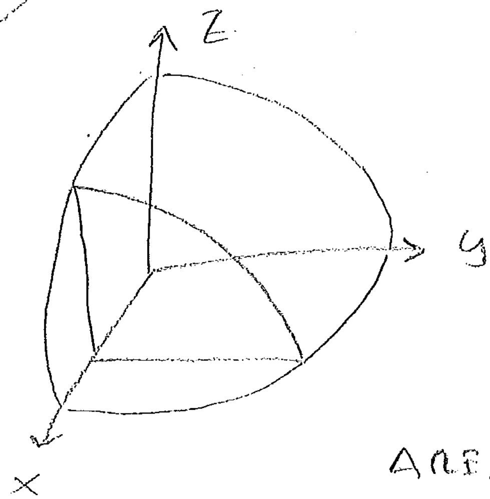
$$\text{VOL} = \int_0^h \pi \frac{x^2 r^2}{h^2}\, dx = \frac{\pi r^2}{h^2} \left(\frac{x^3}{3} \right) \Big|_0^h = h r^2 \cdot \frac{\pi}{3}$$

NOTE: h IS CONSTANT

EXAMPLE: SPHERE OF RADIUS r



← WE'LL COMPUTE THIS AND MULTIPLY BY 8



FIX A VALUE OF X

$$x^2+y^2+z^2=r^2$$

$$\underbrace{y^2+z^2}_{\text{CIRCLE!}}=\underbrace{r^2-x^2}_{\text{CONSTANT}}$$

RADIUS $=\sqrt{r^2-x^2}$

AREA $=(r^2-x^2)\cdot\pi$ $\left(\cdot\frac{1}{4}\right.$ AS WE ARE ONLY TAKING

$\left.\right)$

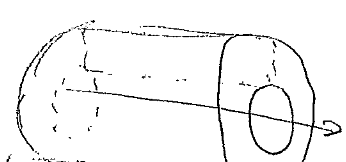
SO VOL OF OCTANT =

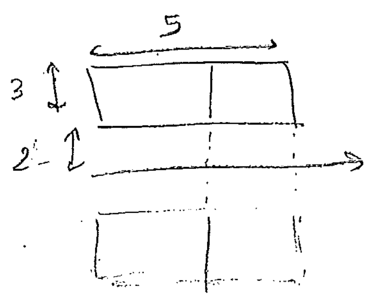
$$\int_0^r \frac{(r^2-x^2)\pi}{4}\,dx=\frac{\pi}{4}\left(xr^2-\frac{x^3}{3}\right)\Bigg|_0^r=\frac{\pi}{4}\cdot\frac{2}{3}r^3=\frac{\pi}{6}r^3$$

VOL OF SPHERE $8\cdot\frac{\pi}{6}r^3=\frac{4}{3}\pi r^3$

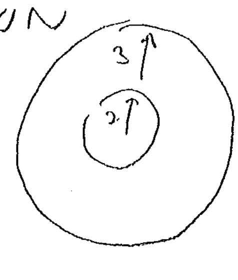
EXAMPLE: REVOLVING A REGION

WE CONSTRUCT AN HOLLOW CYLINDER BY REVOLVING A RECTANGLE AROUND THE X-AXIS





SLICE: A CIRCULAR CROWN



AREA: $5^2\cdot\pi-2^2\cdot\pi=21\pi$