EXAMPLE: LET'S TRY USING THE DEFINITION TO COMPUTE

$$\int_0^4 (x^2-3x)\,dx$$

- x^2-3x IS CONTINUOUS, SO WE KNOW THE INTEGRAL EXISTS, AND MOREOVER WE CAN PICK A RIEMANN SUM OF OUR CHOOSING TO COMPUTE IT. WE'LL USE RIGHT ENDPOINTS.

  – $a=0$, $b=4$, $\Delta x = 4/m$, $x_i = \frac{4i}{m}$

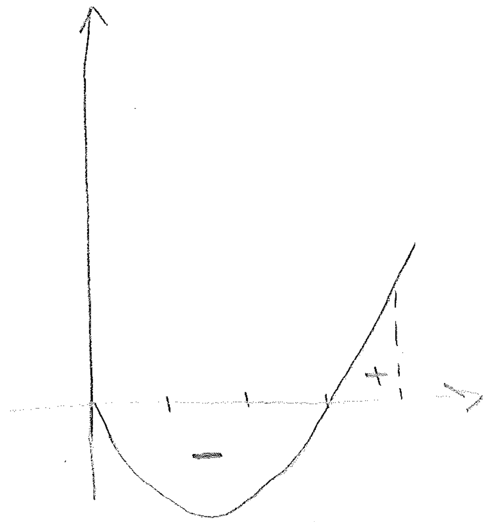
  – $R_m = \sum_{i=1}^{m} f(x_i)\Delta x = \sum_{i=1}^{m}\left(\frac{16i^2}{m^2} - \frac{12i}{m}\right)\frac{4}{m}$

$$= \sum_{i=1}^{m}\frac{64i^2}{m^3} - \sum_{i=1}^{m}\frac{48i}{m^2} = \frac{64}{m^3}\frac{m(m+1)(2m+1)}{6} - \frac{48}{m^2}\frac{m(m+1)}{2}$$

- TAKING THE LIMIT FOR $m\to\infty$ WE GET

$$\int_0^4 f(x)\,dx = \lim_{m\to\infty} R_m = \lim_{m\to\infty}\frac{64}{m^3}\frac{m(m+1)(2m+1)}{6} - \lim_{m\to\infty}\frac{48}{m^2}\frac{m(m+1)}{2} = \frac{64}{3} - 24 = -\frac{8}{3}$$

- THIS WAS NOT REALLY EASY! ALSO, THE ANSWER IS A NEGATIVE NUMBER; LET'S PLOT THE FUNCTION:

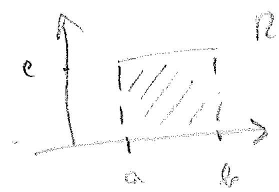


THE "NEGATIVE AREA" OUTWEIGHTS THE POSITIVE.

CAN WE COMPUTE SOME INTEGRALS JUST BY USING GEOMETRY?

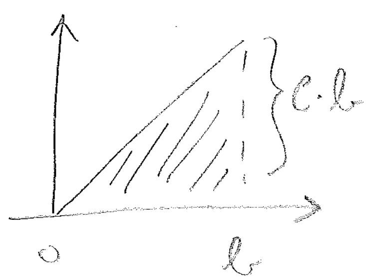
- $\int_a^b c\,dx$ c CONSTANT

  $= (b-a)\cdot c$



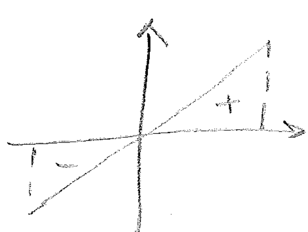
RECTANGLE, AREA IS $(b-a)\cdot c$

- $\int_0^b cx\,dx = \frac{cb^2}{2}$



SQUARE TRIANGLE, AREA $\frac{cb^2}{2}$

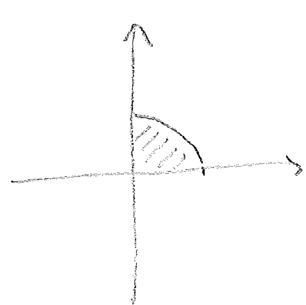
- $\int_{-b}^{b} cx\,dx = 0$



POSITIVE AREA CANCELS OUT NEGATIVE AREA

- $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$

  ↑ FORMULA OF A CIRCLE!



QUARTER OF A CIRCLE, AREA IS $\frac{\pi}{4}$

JUST LIKE SUMMATIONS, INTEGRALS HAVE SOME RELEVANT ARITHMETIC PROPERTIES.

- $\int_a^a f(x)\,dx = 0$ $\left(\text{AS } \Delta_x = 0\right)$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ $\left(\text{AS } \Delta_x \text{ CHANGES FROM } \frac{b-a}{n} \text{ TO } \frac{a-b}{n}\right)$
- $\int_a^b c\,dx = c(b-a)$

$\boxed{* \ f(x), g(x) \ \underline{\text{INTEGRABLE}}}$

- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

SOME ULTERIOR PROPERTIES ARE MORE CLOSELY TIED TO THE GEOMETRIC MEANING OF INTEGRAL

- IF $c \in (a,b)$ THEN

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

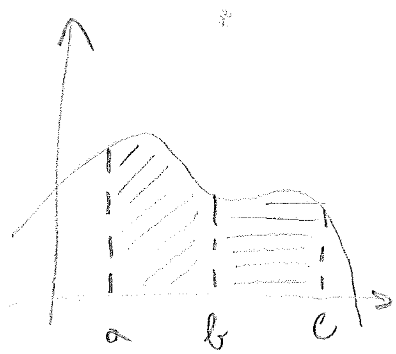
(CUT-AND-PASTE)

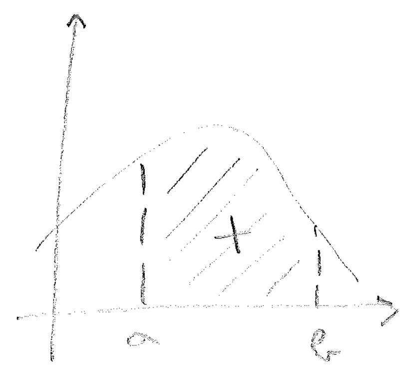
- IF $f(x) \geq 0$ ON $[a, b]$ THEN $\int_a^b f(x)\,dx \geq 0$
- IF $m \leq f(x) \leq M$ ON $[a, b]$ THEN

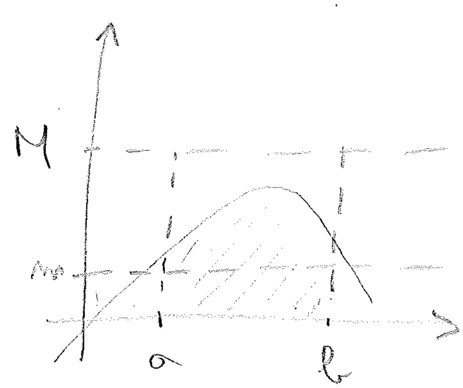
$$m(a-b) \leq \int_a^b f(x)\,dx \leq M(a-b)$$

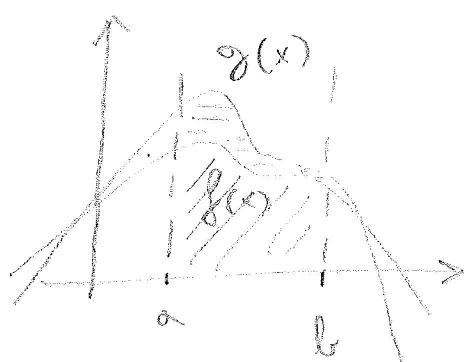
- IF $f(x) \leq g(x)$ ON $[a, b]$ THEN

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$









WE CAN USE THESE FOR ESTIMATES: CONSIDER

$$\int_1^4 \sqrt{x}\,dx$$

$\sqrt{x}$ IS INCREASING, MIN $= 1$ AT $x = 1$, MAX $= 2$ AT $x = 4$

SO

$$3 = 1 \cdot (4-1) \leq \int_1^4 \sqrt{x}\,dx \leq 2 \cdot (4-1) = 6$$

ACTUA VALUE $\frac{14}{3} = 4.\bar{6}$

FOR NOW, IT FEELS LIKE WE'RE TRYING TO ASSEMBLE A CUPBOARD WITH A STONE AND SOME NAILS...  WE NEED SOMETHING BETTER: A HAMMER!

THE FUNDAMENTAL THEOREM OF CALCULUS

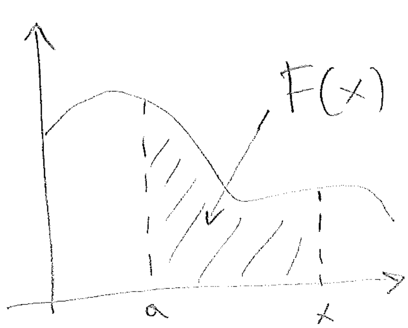
THEOREM (F.T.C., PART 1)

f CONTINUOUS ON $[a,b]$, DEFINE

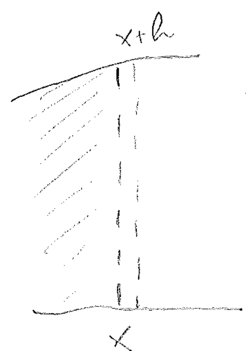
$$F(x) = \int_a^x f(t)\,dt \quad \text{, THEN}$$

$$\frac{d}{dx} F(x) = f(x)$$

- WHY?



MEASURES THE AREA UNDER $f(x)$ FROM a TO x; WHAT IS ITS RATE OF CHANGE?



FOR SMALL h, $f(x+h) \approx f(x)$, SO THE DIFFERENCE IN AREA $\approx h f(x)$

So $$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(x)\,dx - \int_a^{x} f(x)\,dx}{h} =$$

SHOULD BE MADE MORE PRECISE

$$\lim_{h \to 0} \frac{\int_x^{x+h} f(x)\,dx}{h} = \lim_{h \to 0} \frac{h f(x)}{h} = f(x)!$$

SO ROUGHLY STATING,

"DIFFERENTIATING UNDOES INTEGRATING"

EXAMPLE:

- $F(x) = \int_0^x \cos T\,dT$, THEN $F'(x) = \cos(x)$
- $F(x) = \int_1^x (T^2 + \sqrt{T+1})\,dT$, THEN $F'(x) = x^2 + \sqrt{x+1}$
- $F(x) = \int_1^{x^2} \cos T\,dT$, FIND $F'(x)$

CHAIN RULE: $F(x) = G(x^2)$, $G(x) = \int_1^x \cos T\,dT$

$$\frac{d}{dx} G(x^2) = \frac{d}{dx} x^2 \cdot \frac{d}{dx} G(x^2) = 2x \cos x^2$$

THIS IS STILL NOT SOLVING OUR MAIN PROBLEM OF COMPUTING INTEGRALS, THOUGH!