EX: WHY $\log|x| + C$ AND NOT $\log x + C$ AS ANTIDERIVATIVE OF $\frac{1}{x}$?

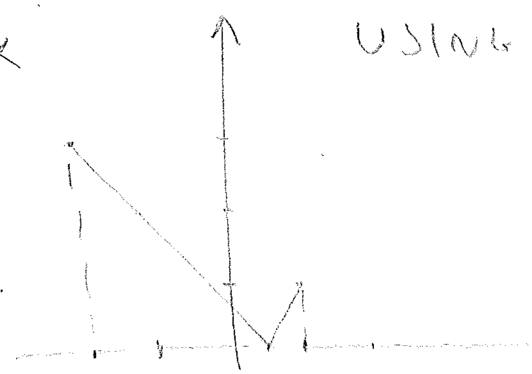
$\log x$ ONLY EXISTS FOR POSITIVE $x$,

$\frac{1}{x}$ FOR ALL $x \neq 0$!

VERIFY: $\log|x| = \begin{cases} \log x & x > 0 \\ \text{UNDEFINED} & x = 0 \\ \log(-x) & x < 0 \end{cases}$

SO $\frac{d}{dx} \log|x| = \begin{cases} \frac{1}{x} & x > 0 \\ (-1)\frac{-1}{x} = \frac{1}{x} & x < 0 \text{ (CHAIN RULE)} \end{cases}$

EX: $\int_{-2}^{1} |2x-1|\, dx$ USING THE FTC



WE SPLIT IN TWO SIMPLER PARTS

$$\int_{-2}^{1} |2x-1|\, dx = \int_{-2}^{\frac{1}{2}} \underbrace{|2x-1|}_{\leq 0} dx + \int_{\frac{1}{2}}^{1} \underbrace{|2x-1|}_{\geq 0} dx$$

$$= \int_{-2}^{\frac{1}{2}} -2x+1\, dx + \int_{\frac{1}{2}}^{1} 2x-1\, dx = \left[x - x^2\right]_{-2}^{\frac{1}{2}} + \left[x^2 - x\right]_{\frac{1}{2}}^{1}$$

$$= \left(\left(\tfrac{1}{2} - \tfrac{1}{4}\right) - (-2-4)\right) + \left((1-1) - \left(\tfrac{1}{4} - \tfrac{1}{2}\right)\right) = \frac{1}{4} + 6 + \frac{1}{4} = \frac{13}{2}$$

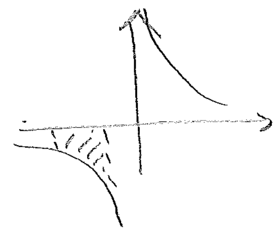
SOME MORE EXAMPLES:

- $\displaystyle\int_{-2}^{-1} \frac{1}{x}\,dx \qquad f(x) = \frac{1}{x}, \quad F(x) = \log|x|$

SO $\displaystyle\int_{-2}^{-1} \frac{1}{x}\,dx = \log|x| \Big|_{-2}^{-1} = \log 1 - \log 2$

$= 0 - \log 2$

NEGATIVE AS EXPECTED



- A <u>NON-EXAMPLE</u>: $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$

WE MIGHT BE TEPTED TO SAY $F(x) = -\frac{1}{x}$,

$\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = -1 - 1 = -2$

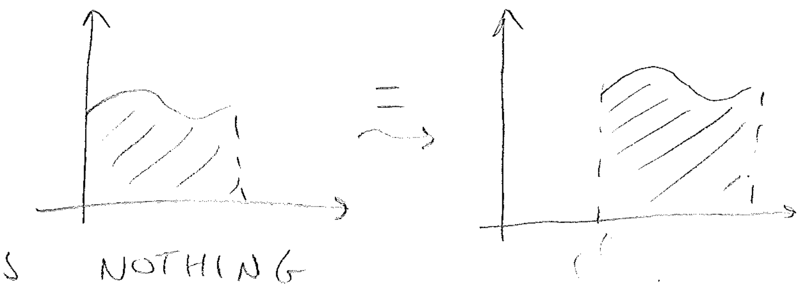
BUT! THIS IS <u>WRONG</u>. $\frac{1}{x^2}$ IS <u>NOT CONTINUOUS</u> AT 0, SO WE CANNOT APPLY THE FTC! IN FACT, WE CANNOT EVEN SAY WHETER IT IS DEFINED. IN FACT, IT <u>IS NOT</u>, AS WE'LL SEE LATER IN THE SEMESTER.

# LINEAR SUBSTITUTION

WHAT HAPPENS TO THE INTEGRAL IF WE COMPOSE BY A LINEAR FUNCTION?

TWO WAYS TO SEE IT:

1) <u>WITH AREAS</u>:



SHIFTING DOES NOTHING

$$\int_a^b f(x+e)\,dx = \int_{a+e}^{b+e} f(x)\,dx$$

RESCALING CHANGES THE RECTANGLE'S WIDTH

$$\int_a^b f(rx)\,dx = \frac{1}{r}\int_{ra}^{rb} f(x)\,dx$$ SO

IN TOTAL $$\int_a^b f(rx+e)\,dx = \frac{1}{r}\int_{ra+e}^{rb+e} f(x)\,dx$$

2) <u>WITH THE FTC</u>: IF $\frac{d}{dx}F(x) = f(x)$

THEN $\frac{d}{dx}F(x+e) = f(x+e)$, $\frac{d}{dx}\frac{F(rx)}{r} = f(rx)$,

$$\frac{d}{dx}\frac{F(rx+e)}{r} = f(rx+e)$$

EX: $\int_1^2 e^{2x-1}\,dx \qquad r=2,\ c=-1,\ f(x)=e^x$

$$\parallel$$

$$\frac{F(2x-1)}{2}\Big|_1^2 = \frac{e^{2x-1}}{2}\Big|_1^2 = \frac{e^3-e}{2}$$

OR

$$ra+c = 2-1 = 1$$

$$rb+c = 4-1 = 3$$

$$\int_1^2 e^{2x-1}\,dx = \frac{1}{2}\int_1^3 e^x\,dx = \frac{e^3-e}{2}$$

NOTE: $\displaystyle\int_1^2 e^{2x-1}\,dx \neq \int_0^1 e^{2x}\,dx$

CORRECT $\displaystyle\int_1^2 e^{2x-1}\,dx = \int_1^2 e^{2(x-\frac{1}{2})}\,dx = \int_{\frac{1}{2}}^{\frac{3}{2}} e^{2x}\,dx$ ,

$$\int_{\frac{1}{2}}^{\frac{3}{2}} e^{2x}\,dx = \int_1^3 e^x\,dx$$

## ODD AND EVEN FUNCTIONS

DEF: $f(x)$ IS <u>ODD</u> IF $f(-x) = -f(x)$.

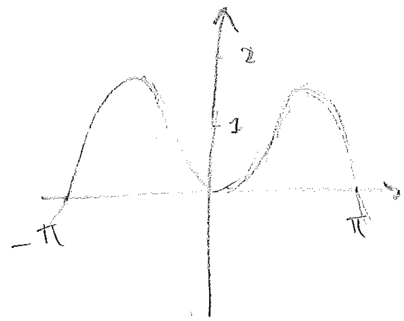
$f(x)$ IS <u>EVEN</u> IF $f(-x) = f(x)$.

Ex: EVEN-DEGREE MONOMIALS ARE EVEN; ODD-DEGREE MONOMIALS ARE ODD.

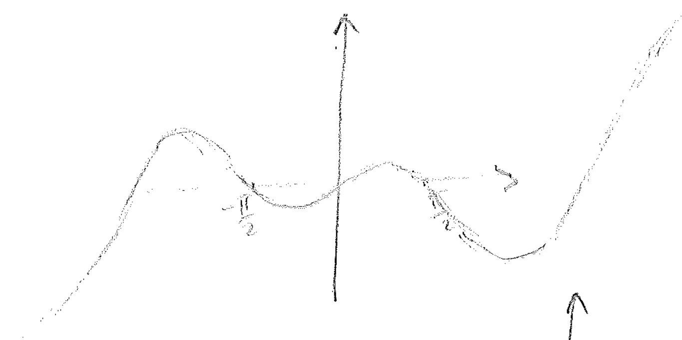
Ex: $\sin(x)$ IS ODD; $\cos(x)$ IS EVEN

<u>NOTE</u>: EVEN AND ODD FUNCTION MULTIPLY <u>LIKE ± SIGNS</u>, <u>NOT LIKE</u> EVEN/ODD NUMBERS.

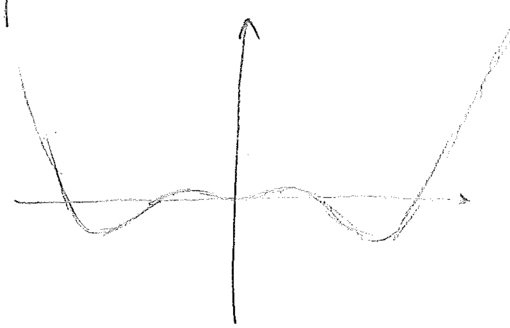
Ex: $x \sin x$ IS <u>EVEN</u>




$x \cos x$ IS <u>ODD</u>




$x^2 \cos x$ IS <u>EVEN</u>



THM: IF $f(x)$ IS EVEN, THEN

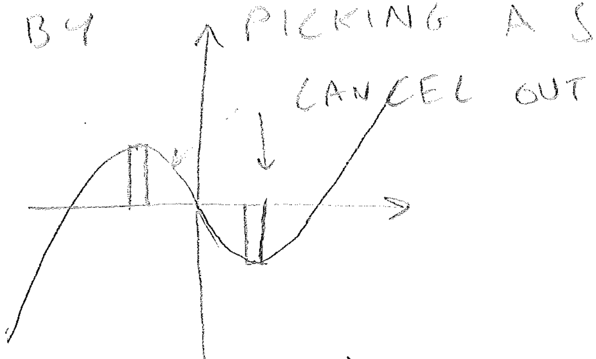
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$ IF $f(x)$ IS ODD,

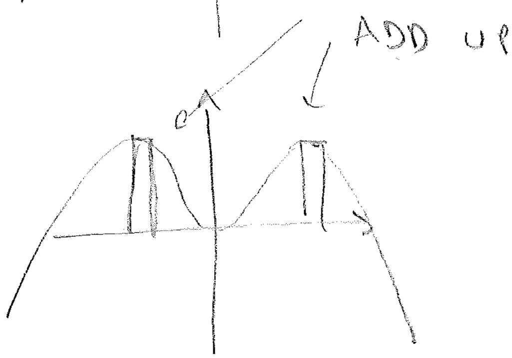
THEN $$\int_{-a}^{a} f(x)\,dx = 0.$$

WHY?

TWO WAYS:

- WE CAN SEE IT WITH THE DEFINITION OF INTEGRAL, BY PICKING A SYMMETRIC R.S.




OR USING LINEAR SUBSTITUTION

$f(x)$ ODD: $\displaystyle\int_{-a}^{0} f(x)\,dx \underset{\substack{\uparrow \\ \text{LINEAR} \\ \text{SUBSTITUTION}}}{=} -\int_{a}^{0} f(-x)\,dx \underset{\substack{\uparrow \\ f(x)\text{ ODD}}}{=} -\int_{a}^{0} -f(x)\,dx$

$\displaystyle = \int_{a}^{0} f(x)\,dx \underset{\substack{\uparrow \\ \text{PROP. OF} \\ \text{INTEGRALS}}}{=} -\int_{0}^{a} f(x)\,dx$

$f(x)$ EVEN: $\displaystyle\int_{-a}^{0} f(x)\,dx \underset{\substack{\uparrow \\ LS}}{=} -\int_{a}^{0} f(-x)\,dx \underset{\substack{\uparrow \\ \text{EVEN}}}{=} -\int_{a}^{0} f(x)\,dx$

$\displaystyle = \int_{0}^{a} f(x)\,dx.$

NOTE: ODD + ODD = ODD; EVEN + EVEN = EVEN

ODD + EVEN = USUALLY NEITHER

EX: FIND $\int_{-10}^{10} \frac{\sin x}{1+x^2+x^4}\, dx$

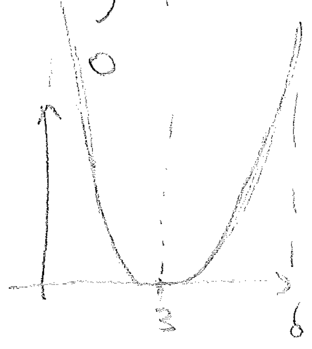
$\sin x$ IS ODD; $1+x^2+x^4$ SUM OF EVEN IS EVEN

$\frac{\sin x}{1+x^2+x^4}$ IS ODD THEN: $\frac{\sin(-x)}{1+(-x)^2+(-x)^4} = \frac{-\sin x}{1+x^2+x^4}$.

THUS $\int_{-10}^{10} \frac{\sin x}{1+x^2+x^4}\, dx = 0$

EX: FIND $\int_0^6 e^{|x-3|}-1 \, dx$

PLOT:



SYMMETRIC AROUND 3!

$\int_0^6 e^{|x-3|}-1\, dx = \int_{-3}^{3} e^{|x|}-1\, dx$ $\qquad$ $e^{|x|}-1$ IS EVEN SO

$\int_{-3}^{3} e^{|x|}-1\, dx = 2\int_0^3 e^{|x|}-1\, dx = 2\int_0^3 e^{x}-1\, d = 2\left[e^x - x\right]_0^3 = \underline{2e^3-8}.$