SOME SUMS YOU SHOULD KNOW:

- FIRST m INTEGERS

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$

WHY?

$$S = 1 + 2 + \dots + m-1 + m$$

$$S = m + m-1 + + 2 + 1$$

$$2S = (m+1) + (m+1) + (m+1) + (m+1) = m \cdot (m+1)$$

- SUM OF POWERS:

$$\sum_{k=0}^{m} r^k = \frac{r^{m+1} - 1}{r-1} \quad \text{FOR ALL } r \neq 1$$

WHY?

$$r^{m+1} - 1 = (r-1)(r^m + r^{m-1} + \dots + 1)$$

- FIRST m SQUARES:

$$\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$$

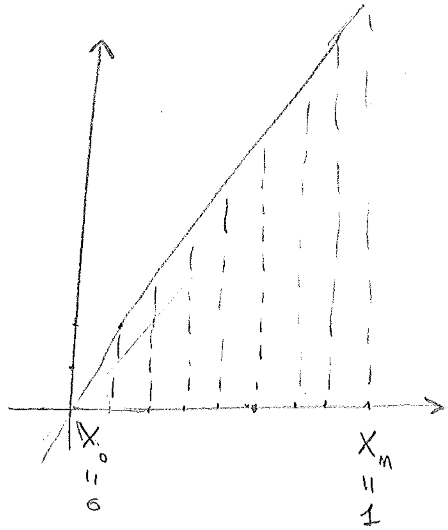
- FIRST m CUBES:

$$\sum_{k=1}^{m} k^3 = \frac{m^2(m+1)^2}{4}$$

PROVEN BY INDUCTION

LET'S TRY COMPUTING A SIMPLE AREA

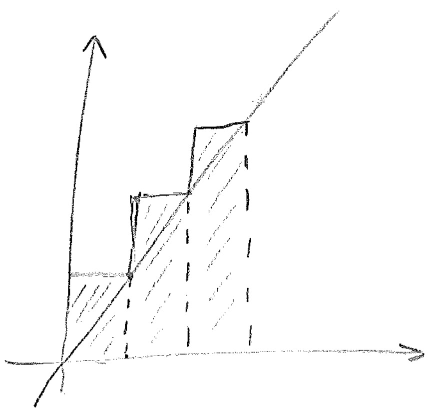
$f(x) = 2x$

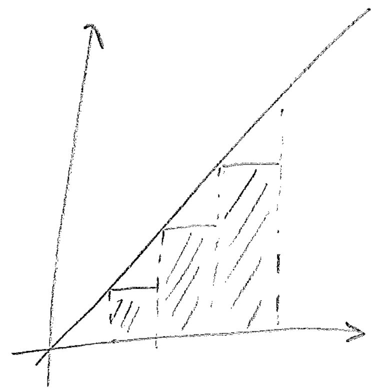


DIVIDE $[0, 1]$ IN n EQUAL SEGMENTS $[x_0, x_1] \dots [x_{n-1}, x_n]$ OF WIDTH

$$\Delta_x = \frac{1}{n}$$

RECTANGLES GIVEN BY RIGHT ENDPOINTS WILL OVERESTIMATE THE AREA, LEFT ENDPOINTS WILL UNDERESTIMATE IT





Q: UNDER WHICH CONDITION RIGHT/LEFT ENDPOINTS OVERESTIMATE/UNDERESTIMATE THE AREA?

A:

	$f(x)$ INCREASING	$f(x)$ DECREASING
LEFT	UNDER	OVER
RIGHT	OVER	UNDER

BACK TO THE SUM: RIGHT ENDPOINTS

- X-ORDINATES $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} = 1$
- HEIGHT OF i-TH RECTANGLE IS $f\left(\frac{i}{n}\right) = \frac{2i}{n}$
- WIDTH IS $\Delta_x = \frac{1}{n}$

• TOTAL AREA IS $R_m = \sum_{i=1}^{m} f(x_i)\Delta_x =$

$$= \sum_{i=1}^{m} \frac{2i}{m} \cdot \frac{1}{m} = \sum_{i=1}^{m} \frac{2i}{m^2} = \frac{2}{m^2}\sum_{i=1}^{m} i =$$

$$= \frac{2}{m^2} \frac{m(m+1)}{2} = \frac{m+1}{m}$$

• THE LIMIT FOR $m \to \infty$ IS

$\lim_{m\to\infty} \frac{m+1}{m} = 1$ SO AREA ≤ 1

LEFT ENDPOINTS:

• HEIGHT OF i-TH RECTANGLE IS $f\left(\frac{i-1}{m}\right) = \frac{2(i-1)}{m}$

• TOTAL AREA IS $L_m = \sum_{i=1}^{m} f\left(\frac{i-1}{m}\right)\Delta_x =$

$$\sum_{i=1}^{m} \frac{2(i-1)}{m} \cdot \frac{1}{m} = \ldots = \frac{2}{m^2}\sum_{i=1}^{m} i =$$

$$\frac{2}{m^2} \frac{(m-1)m}{2} = \frac{m-1}{m}$$

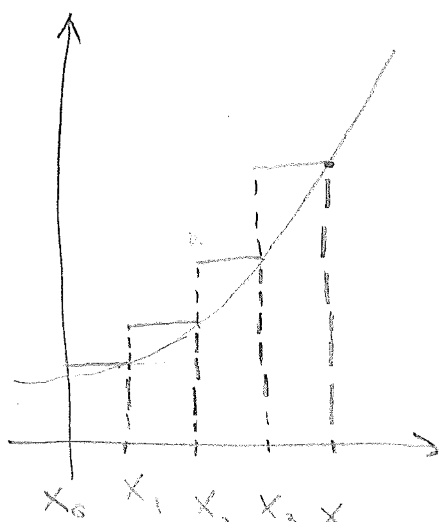
• THE LIMIT FOR $m \to \infty$ IS

$\lim_{m\to\infty} \frac{m-1}{m} = 1$ SO AREA ≥ 1

THUS AREA $= 1$

EXAMPLE: APPROXIMATING AN AREA

IF WE ONLY NEED AN APPROXIMATED RESULT THERE IS NO NEED FOR LIMITS; CONSIDER FOR EXAMPLE THE AREA UNDER $y = e^x$, BETWEEN $x=0$ AND $x=1$; LET'S TRY APPROXIMATING IT WITH 4 RECTANGLES, USING RIGHT ENDPOINTS



$\Delta_x = \frac{1}{4}$ $x_0=0$, $x_1=\frac{1}{4}$, $x_2=\frac{1}{2}$

$x_3=\frac{3}{4}$, $x_4=1$

$$R_4 = f\left(\tfrac{1}{4}\right)\Delta_x + f\left(\tfrac{1}{2}\right)\Delta_x + f\left(\tfrac{3}{4}\right)\Delta_x + f(1)\Delta_x = \frac{1}{4}\left(e^{\frac{1}{4}} + e^{\frac{1}{2}} + e^{\frac{3}{4}} + e\right)$$

$$\simeq 1.94$$

ACTUAL AREA $= e-1 \simeq 1.72$, SO IT'S NOT A TERRIBLE APPROX. LATER IN THE COURSE WE'LL BE ABLE TO GET AN "A PRIORI" ESTIMATE FOR THE ERROR DEPENDING ON # RECTANGLES.

EXAMPLE: DISTANCES

IF A BODY IS MOVING AT CONSTANT VELOCITY THEN

DISTANCE = VELOCITY × TIME

WHICH LOOKS A LOT LIKE

AREA = HEIGHT × WIDTH

SO IT MAKES SENSE TO USE OUR APPROXIMATION TECHNIQUE TO OBTAIN DISTANCES TOO!

SAY WE KNOW THE VELOCITY OF A CAR AT REGULAR INTERVALS

TIME (S)	0	10	20	30	40	50	60
VELOCITY (M/S)	3	7	8	11	13	12	11

WE CAN APPROXIMATE HOW FAR THE CAR WENT!

RIGHT ENDPOINTS:

$$D_R = 7\times 10 + 8\times 10 + 11\times 10 + 13\times 10 + 12\times 10 + 11\times 10$$

$$= (7+8+11+13+12+11)\times 10 = 620 \text{ m}$$

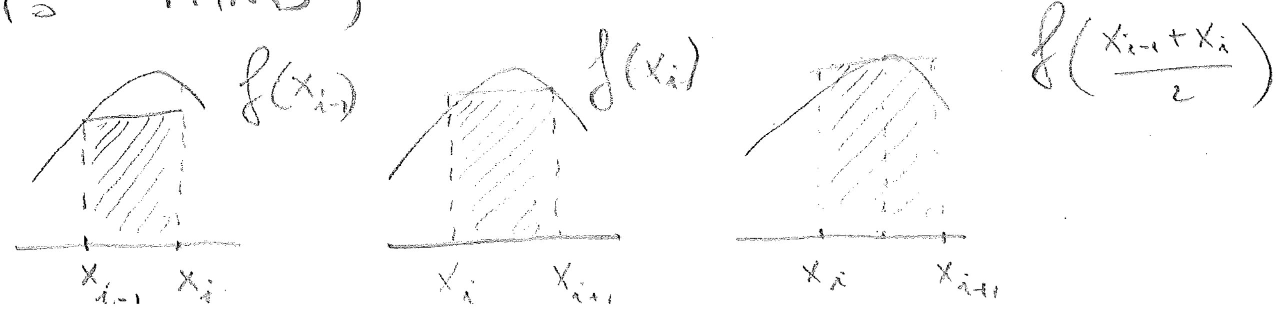
LEFT ENDPOINTS

$$D_L = 3\times 10 + 7\times 10 + 8\times 10 + 11\times 10 + 13\times 10 + 12\times 10 =$$

$$(3+7+8+11+13+12)\times 10 = 540 \text{ m}$$

THE DEFINITE INTEGRAL

WHEN TRYING TO APPROXIMATE THE AREA UNDER $y = f(x)$ THERE ARE AT LEAST 3 CHOICES OF POINTS TO PICK VALUES THAT COME TO MIND;



RESPECTIVELY LEFT ENDPOINTS, RIGHT ENDPOINTS AND MIDPOINTS.

BUT! HOW DO WE KNOW WHICH TO PICK, OR EVEN BETTER, THAT IT DOESN'T MATTER WHEN WE TAKE THE NUMBER OF RECTANGLES TO INFINITY?

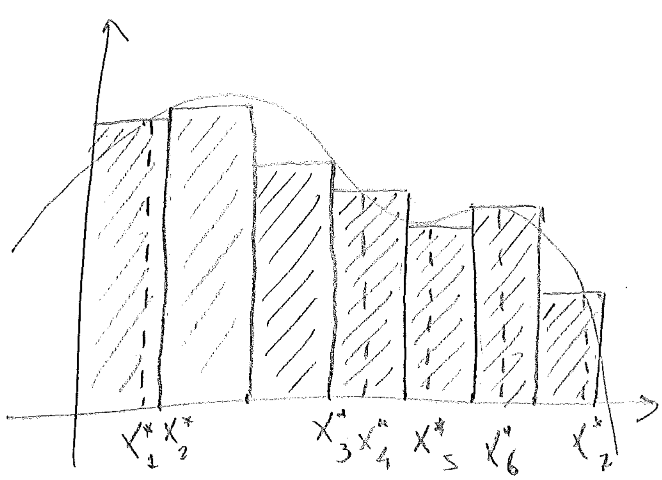
NOTATION

$f(x)$ A FUNCTION DEFINED ON $[a, b]$

$\Delta_x = \dfrac{b-a}{n}$ $\quad x_0 = a, x_1 = a + \Delta_x, x_2 = a + 2\Delta_x, \ldots, x_n = b$

A RIEMANN SUM IS A SUM IN THE FORM

$\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta_x$ WHERE x_i^* IS ANY POINT $x_{i-1} \le x_i^* \le x_i$. SO FOR EXAMPLE THE APPROXIMATION



IS GIVEN BY A RIEMANN SUM

DEFINITION

WE SAY $f(x)$ IS INTEGRABLE ON $[a,b]$ IF FOR ANY CHOICE OF POINTS x_i^* THE LIMIT

$$\lim_{m \to \infty} \sum_{i=1}^{m} f(x_i^*) \Delta_x$$

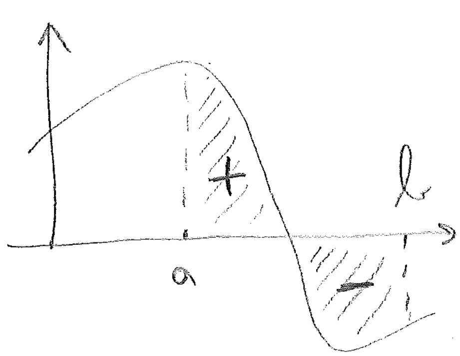
EXISTS AND IS THE SAME. IN THIS CASE WE DEFINE IT TO BE

$$\lim_{m \to \infty} \sum_{i=1}^{m} f(x_i^*) \Delta_x = \int_a^b f(x)\,dx$$

THE DEFINITE INTEGRAL OF $f(x)$ FROM a TO b.

NOTES:

- IF $f(x)$ IS POSITIVE, $\int_a^b f(x)\,dx$ IS THE AREA UNDER THE CURVE $y = f(x)$
- IN GENERAL WE'LL GET A "SIGNED AREA" DEPENDING ON THE SIGN OF $f(x)$

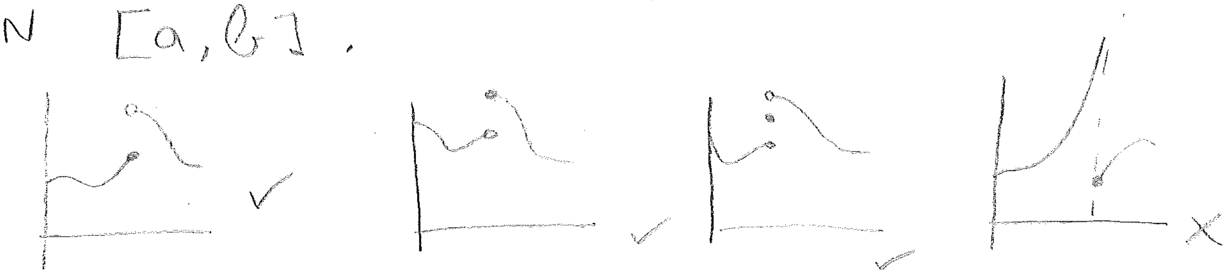


$$\int_a^b f(x)\,dx = \text{AREA+} - \text{AREA-}$$

- $f(x)$ IS CALLED THE INTEGRAND, a, b THE LIMITS OF INTEGRATION
- dx DOES NOT HAVE A MEANING BY ITSELF, ONLY IN FORMULA.
- $\int_a^b f(x)\,dx$ IS A NUMBER, DOES NOT DEPEND ON x

SO, HOW DO WE VERIFY THE CONDITION ON RIEMANN SUMS? WE CAN'T CHECK THEM ALL, RIGHT?

<u>THEOREM</u>: IF $f(x)$ IS CONTINUOUS OR ONLY HAS A FINITE NUMBER OF <u>JUMP DISCONTINUITIES</u>* ON $[a, b]$, THEN $f(x)$ IS INTEGRABLE ON $[a, b]$.

* 

SO, BASICALLY ALL REASONABLE FUNCTIONS WITH NO ASYMPTOTES ARE INTEGRABLE!

Q (HARD!): SHOW THAT $f(x) = \begin{cases} 1 & x \text{ IS RATIONAL} \\ 0 & x \text{ IS NOT} \end{cases}$ IS <u>NOT</u> INTEGRABLE ON $[0, 1]$

A: IF WE TAKE LEFT ENDPOINTS THEN $\sum_{i=1}^{n} f(x_i^*) \Delta_x = \frac{1}{n} \cdot n = 1$, IF WE CHOOSE IRRATIONAL x_i^* WE GET 0, SO THE LIMIT IS NOT UNIQUE