WARM-UP DISCUSS THE I.O.C. OF

- $\sum_{m=0}^{\infty} (\pi_m + 1) x^m$

- $\sum_{m=0}^{\infty} \frac{x^{2m}}{m^2 \cdot 2^m}$

- $|(\pi_m + 1) x^m| \leq |10 x^m|$ SO BY COMPARISON

$\sum_{m=0}^{\infty} (\pi_m + 1) x^m$ CONV. ABS. ON $(-1, 1)$

BY DIV. TEST, $\sum_{m=0}^{\infty} (\pi_m + 1) x^m$ DIVERGES WHEN

$x = \pm 1$

- RATIO TEST $\lim_{m \to \infty} \left| \frac{x^{\cancel{2m}+2}}{(m+1)^2 \, 2^{m+1}} \cdot \frac{m^2 \, \cancel{2^m}}{\cancel{x^{2m}}} \right| =$

$\lim_{m \to \infty} \frac{|x^2|}{2} \cdot \frac{m^2}{(m+1)^2} = \frac{|x^2|}{2}$ $\quad \frac{|x^2|}{2} < 1$

$\sim$ $x < \sqrt{2}$. AT $x = \pm\sqrt{2}$ CONV. ABS.

SO I.O.C. IS $[-\sqrt{2}, \sqrt{2}]$

<u>THM:</u> SUPPOSE R.O.C. OF $\sum_{m=0}^{\infty} a_m (x-c)^m$ IS R.

THEN R.O.C. OF $\sum_{m=0}^{\infty} a_m (x-c)^{m \cdot m}$ IS $\sqrt[m]{R}$

<u>EVALUATION</u> : HTTPS://EVAL.CTLT.UBC.CA/SCIENCE

THM: SUPPOSE $\sum_{m=0}^{\infty} a_m (x-c)^m = f(x)$ HAS R.O.C. R. THEN

- $F(x) = \int_c^x f(\tau)\,d\tau = \sum_{m=1}^{\infty} \frac{a_{m-1}}{m}(x-c)^m$

- $f'(x) = \sum_{m=0}^{\infty} a_{m+1}\cdot(m+1)(x-c)^m$

AND BOTH $F(x), f'(x)$ HAVE R.O.C. R.

NOTE: $\int f(x)\,dx = C + \sum_{m=1}^{\infty} \frac{a_{m-1}}{m}(x-c)^m$

EX: $\log(1+x)$ AS $\sum$

$$\frac{1}{1+x} = \sum_{m=0}^{\infty} (-1)^m x^m \qquad (c=0,\ R=1)$$

$$\log(1+x) = \int_0^x \frac{1}{1+\tau}\,d\tau = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} x^m$$

EX: $\arctan(x)$ AS $\sum$

$$\frac{1}{1+x^2} \underset{\substack{\uparrow \\ \text{PLUG } x^2 \\ \text{IN PREV FORM}}}{=} \sum_{m=0}^{\infty} (-1)^m x^{2m} \qquad (c=0,\ R=1)$$

$$= 1 + 0 - x^2 + 0 + x^4 + 0 - x^6 + \ldots$$

$$\int_0^x \frac{1}{1+\tau^2}\,d\tau = x + 0 - \frac{x^3}{3} + 0 + \frac{x^5}{5} + 0 - \frac{x^7}{7} \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} x^{2m+1}$$

EX: COMPACT FORM FOR $\sum_{m=1}^{\infty} (m-1) x^m$

$$S = x^2 + 2x^3 + 3x^4 + 4x^5 + \ldots$$

IDEA: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$\frac{x^2}{(1-x)^2} = x^2 + 2x^3 + 3x^4 + 4x^5 + \ldots = \sum_{m=1}^{\infty} (m-1) x^m$$

TWO QUESTIONS:

Q1 GIVEN $f(x) = \sum_{m=0}^{\infty} a_m (x-c)^m$, WHAT IS $\frac{d^m f}{dx^m}(c)$?

Q2 IF WE KNOW $\frac{d^m f}{dx^m}(c)$ FOR ALL m, CAN WE RECONSTRUCT $\sum_{m=0}^{\infty} a_m (x-c)^m$?

THM: (TAYLOR SERIES)

- IF $f(x) = \sum_{m=0}^{\infty} a_m (x-c)^m$, THEN $\frac{d^m f}{dx^m}(c) = a_m \cdot m!$

- IF $f(x)$ IS ∞-DIFFERENTIABLE AT c AND "REASONABLE" THEN:

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m f}{dx^m}(c) (x-c)^m$$

ON THE I.O.C. OF THE POWER SERIES.

(NOTE: $\frac{d^0}{dx^0} f(x) = f(x)$ BY DEF)

AND THIS EXPRESSION IS <u>UNIQUE</u>.

<u>EXAMPLE</u>: e^x AT $c=0$

WE HAVE $\frac{d^m e^x}{dx^m} = e^x$ SO $\frac{d^m e^x}{dx^m}(0) = 1$

THUS $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$ AND $R = \infty$ BY RATIO TEST.

<u>EXAMPLE</u>: $\cos x, \sin x$ AT $c=0$

$\left\{ \frac{d^m \sin x}{dx^m}(0) \right\} = \sin(0), \cos(0), -\sin(0), -\cos(0), \sin(0), \cos(0), \ldots$

$= 0, 1, 0, -1, 0, 1, 0, -1, \ldots$ SO

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$ (ODD POWERS, ALT. SIGN)

$R = \infty$ BY COMP WITH e^x OR RATIO TEST THM.

$\left\{ \frac{d^m \cos x}{dx^m}(0) \right\} = 1, 0, -1, 0, 1, 0, -1 \ldots$ SO

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}$$ (EVEN POWERS, ALT. SIGN)

$R = \infty$, SAME REASON

AN UNREASONABLE FUNCTION

SET $f(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$

THIS FUNCTION IS DIFFERENTIABLE ∞ MANY TIMES AT $c=0$. EXAMPLE:

$$\frac{df}{dx}(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} =$$

$$\lim_{h \to 0} \begin{cases} 0 & \text{IF } h < 0 \\ \underbrace{\frac{e^{-\frac{1}{h}}}{h}}_{\text{EXP BEATS POLY}} & \text{IF } h > 0 \end{cases} = 0$$

AND WE HAVE $\frac{d^m f}{dx^m}(0) = 0$ FOR ALL m. THEN

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m f}{dx^m}(0) x^m = 0 \quad \text{HAS R.O.C. } \infty$$

BUT $f(x) \neq 0$ IN ANY NON-TRIVIAL NEIGHBORHOOD OF $c=0$!