WARM-UP (HTTPS://EVAL.CTLT.UBC.CA/SCIENCE)

DISCUSS CONVERGENCE OF

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x+2)^n$ (FUNCTION OF X!)

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{(n^2+1)(n!)^2}$

- RATIO TEST $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x+2|^{n+1}}{|x+2|^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$

$$= \lim_{n\to\infty} |x+2| \cdot \frac{1}{\sqrt{1+\frac{1}{n}}} = |x+2|$$

SO IT CONVERGES WHEN $|x+2|<1$, THAT IS $-3<x<-1$, DIVERGES WHEN $x<-3$ OR $x>-1$. WHEN $x=-3$

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES; WHEN $x=-1$

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\cdot(1)^n}{\sqrt{n}}$ CONV BY ALT TEST. SO $\sum$ CONV.

ON $(-3,-1]$

- RATIO TEST $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+2)!}{2n!} \cdot \frac{n^2+1}{(n+1)^2+1} \cdot \frac{(n!)^2}{((n+1)!)^2}$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)\cdots 2}}{\cancel{(2n)\cdots 2}} \cdot \frac{\cancel{n^2\cdots 2^2}}{(n+1)^2\cancel{n^2\cdots 2^2}} \cdot \frac{\overbrace{n^2+1}^{\to 1}}{(n+1)^2+1} =$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4 \quad \text{SO IT DIVERGES.}$$

POWER SERIES

CONSIDER A NUMBER x WITH $|x|<1$.

THEN

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

AS $\sum_{m=0}^{\infty} x^m$ IS GEOMETRIC WITH RATIO x AND STARTING VALUE 1 ($x^0 = 1$ FOR ALL x BY CONVENTION)

WE CAN THINK OF THIS AS AN EQUALITY OF FUNCTIONS!

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

WHY DO WE CARE?

IDEA: MAYBE WE CAN APPLY OPERATIONS TERM BY TERM?

$$\frac{d}{dx}\frac{1}{1-x} \overset{?}{=} \begin{array}{ccccccccccc} 1 & + & x & + & x^2 & + & x^3 & + \ldots & + & x^m & + \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\ 0 & + & 1 & + & 2x & + & 3x^2 & + \ldots & + & mx^m & + \ldots \end{array}$$

$$\int \frac{1}{1-x}\,dx = \begin{array}{ccccccccccc} 1 & + & x & + & x^2 & + & x^3 & + \ldots & + & x^m & + \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\ x & + & \frac{x^2}{2} & + & \frac{x^3}{3} & + & \frac{x^4}{4} & + \ldots & + & \frac{x^m}{m} & + \ldots + C \end{array}$$

$$\parallel$$

$-\log|1-x| + C$! BUT THEN ...

$$\log|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{m-1}x^m}{m} + \ldots$$

CONV AT $x=1$ ... SO $\log(2) \overset{?}{=} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}$

DEF:

A SERIES IN THE FORM

$$\sum_{n=0}^{\infty} a_n x^n \qquad (x^0 = 1 \text{ BY DEF})$$

IS A POWER SERIES CENTERED AT 0.

(SAME FOR $\sum_{n=N_0 \geq 0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} a_n x^{2n}$, ...)

A SERIES IN THE FORM

$$\sum_{n=0}^{\infty} a_n (x-c)^n \qquad ((x-c)^0 = 1 \text{ BY DEF})$$

IS A POWER SERIES CENTERED AT c.

(SAME FOR $\sum_{n=N_0}^{\infty} a_n (x-c)^n$, $\sum_{n=0}^{\infty} a_n (x-c)^{2n}$, ...)

THM:

GIVEN A P.S. $\sum_{n=0}^{\infty} a_n (x-c)^n$, THERE ARE THREE POSSIBILITIES:

- THERE EXISTS $R > 0$ S.T. $\sum_{n=0}^{\infty} a_n (x-c)^n$ CONVERGES FOR ANY $|x-c| < R$, DIVERGES FOR $|x-c| > R$. IT CAN DO EITHER AT $x = c \pm R$.
- THE P.S. $\sum_{n=0}^{\infty} a_n (x-c)^n$ CONV. EVERYWHERE ($R = \infty$)
- THE P.S. $\sum_{n=0}^{\infty} a_n (x-c)^n$ ONLY CONV. AT $x = c$ ($R = 0$)

R IS CALLED THE RADIUS OF CONVERGENCE OF $\sum_{n=0}^{\infty} a_n (x-c)^n$.

THE INTERVAL WHERE $\sum_{n=0}^{\infty} a_n (x-c)^n$ CONVERGES IS CALLED INTERVAL OF CONVERGENCE.

ON THIS INTERVAL WE CAN THINK OF $\sum_{n=0}^{\infty} a_n (x-c)^n$ AS A FUNCTION $f(x)$.

EX: FIND THE I.O.C. OF $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

RATIO TEST: $\left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = \left|\frac{x^{n+1}}{x^n}\right| \cdot \frac{n}{n+1} = |x| \frac{n}{n+1}$

SO $\lim_{n \to \infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = |x| \qquad R = 1$

AT $x = 1$ $\quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1^n}{n}$ CONV BY ALT. TEST

AT $x = -1$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$ DIVERGES

SO I.O.C. $= (-1, 1]$, $\; -1 < x \leq 1$

EX: I.O.C. OF $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

RATIO TEST: $\left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = |x| \cdot \frac{n!}{(n+1)!} = \frac{|x|}{n+1}$

SO $\lim_{n \to \infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = \lim_{n \to \infty} \frac{|x|}{n+1} = 0 \qquad R = \infty!$

EX: I.O.C. OF $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{2^n (n+2)^2}$ ← $c = 1$

RATIO TEST: $\left| \frac{a_{n+1}(x-1)^{n+1}}{a_n (x-1)^n} \right| = \frac{2^n}{2^{n+1}} \frac{(n+2)^2}{(n+3)^2} |x-1|$ $\quad \frac{2^n}{2^{n+1}} = \frac{1}{2}$, $\frac{(n+2)^2}{(n+3)^2} \nearrow 1$

SO $\lim_{n\to\infty} \left| \frac{a_{n+1}(x-1)^{n+1}}{a_n (x-1)^n} \right| = \frac{|x-1|}{2}$ $\quad R = 2$

AT $x = 3$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n (n+2)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)^2}$

CONV ABS BY COMP WITH $\sum \frac{1}{(n+2)^2}$

AT $x = -1$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n (-2)^n}{2^n (n+2)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$

SAME. SO I.O.C. IS $[-1, 3]$

$-1 \le x \le 3$.

THM: IF $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ THEN THE R.O.C. OF $\sum_{n=1}^{\infty} a_n (x-c)^n$ IS $\frac{1}{L}$.

EX: $\sum_{n=1}^{\infty} (\pi_n + 1)(x-2)^n$

$\lim_{n\to\infty} \frac{\pi_{n+1}+1}{\pi_n}$ DNE SO RATIO TEST FAILS.

BUT! $|(\pi_n+1)(x-2)^n| \le |10(x-2)^n|$ WHICH CONV. ABS. ON $(1, 3)$, SO $R = 1$. THE P.S. DIVERGES AT $x = 1, 3$ BY DIV. TEST.