Warm-up

Discuss convergence of

1. $\sum_{n=5}^{\infty} \frac{\log(n)^{10}}{n^2}$
2. $\sum_{n=3}^{\infty} \frac{1}{n\sqrt{3\log(n)+2}}$

1. We know that $\log(n)^{10}$ is "very small" compared to n^2, but comparing with $\frac{1}{n^2}$ would be inconclusive. But!

   Every power of logarithm loses to every power of x.

   So we can try comparing it with a slightly smaller negative power of n, as long as it's big enough that the series converges. Let's try $b_n = \frac{1}{n^{3/2}}$.

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\log(n)^{10} n^{3/2}}{n^2} = \lim_{n\to\infty} \frac{\log(n)^{10}}{\sqrt{n}} = 0$$

   So by limit comparison with $\sum_{n=5}^{\infty} \frac{1}{n^{3/2}}$ our series converges.

   Note 1. This shows that sometimes we are forced to use the weaker part of the limit comparison theorem, as sometimes there are no "simple" sequences going to zero at the same rate as ours.

2. Note first that comparing with $\frac{1}{n}$ would give us $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ which is inconclusive as the series of $\frac{1}{n}$ diverges. To show that something diverges we have to use the stronger statement of the limit comparison theorem.

   We have $a_n \sim \frac{1}{n\sqrt{\log(n)}}$, so we try comparing with $b_n = \frac{1}{n\sqrt{\log(n)}}$. We get

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n\sqrt{\log(n)}}{n\sqrt{3\log(n)+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{3+2\log(n)^{-1}}} = \frac{1}{\sqrt{3}}$$

   so by limit comparison our series diverges as $\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\log(n)}}$ diverges by the integral test.

The alternating series test

A sequence b_n is *alternating* if the signs of two consecutive terms are always different (or zero). In particular, b_n is alternating if there is a sequence $a_n \geq 0$ such that either $b_n = (-1)^n a_n$ or $b_n = (-1)^{n-1}(a_n)$, depending on the sign of the first term.

Alternating series (i.e. series where the underlying sequence in alternating) have the peculiarity that consecutive terms "cancel out", making for a very permissive convergence criterion:

Theorem 2 (Alternating series test)**.** *Let b_n be an alternating sequence, $a_n = |b_n|$ the corresponding non-negative sequence. Suppose that:*

- *a_n is decreasing, i.e. $a_{n+1} \leq a_n$ for all n.*
- $\lim_{n\to\infty} a_n = 0$

Then the series $\sum_{n=1}^{\infty} b_n$ converges.

Moreover, call S the value of the series. Then $S - S_N$ is between 0 and b_{N+1}.

Note 3. The theorem is only really relevant when $a_n > 0$ for all n... why?

Note 4. The remainder part of the theorem also tells the sign of $S - S_N$. It's either 0 or the same sign as b_{N+1}.

Note 5. As with the other tests, we can restate the hypotheses asking them to work for all $n \geq N_0$, for some N_0. Be careful though! The remainder part will only work when $N > N_0$.

Example 6. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but what about $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$? Well, $a_n = \frac{1}{n}$ is clearly decreasing, so by the alternating test it must converge. In fact, by the end of the course we will be able to prove that it converges to $\log(2)$.

Now, say we wanted to approximate $\log(2)$ from the left (i.e. with a smaller number), with an error of at most 10^{-3}. How many terms of the sequence do we have to take?

By the theorem

$$\log(2) - S_N = b_{N+1} = \frac{(-1)^N}{N+1}$$

So first we need to have $N+1 \geq 1000$, which yields $N \geq 999$. But when $N = 999$ we have $b_{N+1} = \frac{-1}{1000}$, so the error is negative, which means that S_N is bigger than $\log(2)$, not smaller! Then the first N such that S_N satisfies our conditions as an approximation of $\log(2)$ is $N = 1000$.

Example 7. Consider the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n-\sqrt{n}}$.

1. Show that it converges.
2. Set S to be the value of the series. What is the sign of S?
3. Find an N such that $|S - S_N| \leq 10^{-3}$

1. The sequence is alternating and clearly goes to zero, so we only have to prove that it is decreasing. Set $f(x) = \frac{1}{x-\sqrt{x}}$. Proving that $f(x)$ is decreasing for $x \geq 2$ is equivalent to proving that $g(x) = x - \sqrt{x}$ is increasing. We have $g'(x) = 1 - \frac{1}{2\sqrt{x}}$ which is greater than 0 for $x \geq 2$. So by the alternating test our series converges.

2. To answer the second question, fist set by convention $b_n = \frac{(-1)^{n+1}}{n+1-\sqrt{n+1}}$ so that b_n is exactly the n-th term in the sum. By the remainder part of the theorem, $0 \leq S - S_1 \leq b_2$ (note that S_2 is positive). Adding S_1 to all terms, we get $S_1 \leq S \leq S_1 + S_2$. Now, S_1 is just b_1, so we get $\frac{1}{2-\sqrt{2}} \leq S \leq \frac{1}{2-\sqrt{2}} - \frac{1}{3-\sqrt{3}}$. The terms on the right and left are positive, so S is positive. In fact, this always work! If a series converges by the alternating test then its value is between 0 and the first term.

3. We need $N+1-\sqrt{N+1} \geq 10^3$. Let's begin with a very weak estimate. For $N \geq 3$ we have for sure that $\sqrt{N+1} \leq \frac{N+1}{2}$, so $N+1-\sqrt{N+1} \geq \frac{N+1}{2}$ and $N = 1999$ works. This is a perfectly valid answer. Now, what if we want a sharper solution? Set $u = \sqrt{N+1}$. Then our inequality becomes
$$u^2 - u \geq 10^3$$
solving the quadratic polynomial and taking the positive solution we get
$$u = \frac{1+\sqrt{1+4000}}{2} \leq 33$$
so
$$\sqrt{N+1} \geq 33 \Rightarrow N+1 \geq 1089$$
is the sharp solution to our question.