$$\lim_{n\to\infty} \frac{n+3}{2n+5} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{2+\frac{5}{n}} = \frac{1}{2}$$

THIS LOOKS LIKE WHAT WE WOULD DO FOR A REGULAR LIMIT... AND IN FACT:

THM: IF $\lim_{x\to\infty} f(x) = A$, THEN $\{a_n = f(n)\}$ CONVERGES TO A.

EXAMPLE: $\{a_n = e^{-n}\}$ THEN $\lim_{n\to\infty} a_n = \lim_{x\to\infty} e^{-x} = 0$

NOTE THAT THIS <u>DOES NOT</u> GO TWO WAYS!

EXAMPLE:

$f(x) = \sin(\pi x)$ $\quad \lim_{x\to\infty} \sin(\pi x)$ DNE AS $f(x)$ KEEPS OSCILLATING BETWEEN 1 AND -1.

BUT! $\{a_n = f(n)\}_{n=1}^{\infty} = \{a_n = \sin(\pi n)\}_{n=1}^{\infty}$ <u>DOES</u> CONVERGE AS $\sin(\pi n) = 0$ FOR ALL n!

LIMITS OF SEQUENCES BEHAVE AS WE GENERALLY EXPECT IN TERMS OF THE USUAL OPERATIONS:

THEOREM: IF $\{a_m\}$ CONVERGES TO A AND $\{b_m\}$ CONVERGES TO B THEN:

SEQUENCE	CONVERGES TO
$\{a_m + b_m\}$	$A+B$
$\{a_m - b_m\}$	$A-B$
$\{C a_m\}$	CA
$\{a_m \cdot b_m\}$	$A \cdot B$
* $\left\{\frac{a_m}{b_m}\right\}$	$\frac{A}{B}$

*: PROVIDED $b_m \neq 0$, $B \neq 0$

EXAMPLE: $\left\{-\frac{m+3}{2m+5} + 3e^{-m}\right\}$

$$\lim_{m\to\infty} -\frac{m+3}{2m+5} + 3e^{-m} = \lim_{m\to\infty} 3(e^{-m}) - \left(\frac{m+3}{2m+5}\right) = 3 \cdot 0 - \frac{1}{2}$$

CONV TO 0

CONV TO $\frac{1}{2}$

$= \frac{1}{2}$

ANOTHER POWERFUL TOOL:

THM (SQUEEZE THEOREM):

IF $a_n \leq c_n \leq b_n$ FOR ALL (SUFFICIENTLY LARGE) n, AND

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = L$$

THEN $\lim_{n \to \infty} c_n = L$.

EXAMPLE: $\lim_{n \to \infty} 1 + \frac{\pi_n}{n}$

WHERE π_n = n-TH DECIMAL DIGIT OF π

BY DEFINITION $\pi_n \leq 9$ AND $\pi_n \geq 0$

SO

$$1 + \frac{\pi_n}{n} \leq 1 + \frac{9}{n} \quad \text{AND} \quad 1 + \frac{\pi_n}{n} \geq 1,$$

THEN

$$\underbrace{1}_{= a_n} \leq 1 + \frac{\pi_n}{n} \leq \underbrace{1 + \frac{9}{n}}_{= b_n}$$

NOW $\lim_{n \to \infty} a_n = 1$, $\lim_{n \to \infty} b_n = 1$ SO

$$\lim_{n \to \infty} 1 + \frac{\pi_n}{n} = 1.$$

ONE LAST "ARITHMETHIC" PROPERTY

THM: SUPPOSE $\lim_{n\to\infty} a_n = L$, AND $g(x)$ CONTINUOUS AT $x = L$. THEN

$$\lim_{n\to\infty} g(a_n) = g(L).$$

EXAMPLE: $\lim_{n\to\infty} \arcsin\left(\frac{n+3}{2n+5}\right) = \lim_{n\to\infty} \arcsin(a_n)$

WHERE $a_n = \frac{n+3}{2n+5}$. WE KNOW THAT

$\lim_{n\to\infty} a_n = \frac{1}{2}$, SO $\lim_{n\to\infty} \arcsin\left(\frac{n+3}{2n+5}\right) =$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

SERIES

IF WE HAVE A SEQUENCE $\{a_m\}_{m=1}^{\infty}$

ANOTHER IMPORTANT OBJECT TO CONSIDER IS THE SERIES $\sum_{m=1}^{\infty} a_m$. ROUGHLY SPEAKING $\sum_{m=1}^{\infty} a_m$ IS THE SUM OF ALL THE NUMBERS a_m. WHAT DOES IT EVEN MEAN?

EXAMPLE: (DECIMAL EXPANSION)

TAKE $\frac{1}{3} = 0.3333\ldots$ THIS EXPANSION LITERALLY MEANS THAT

$$\frac{1}{3} = 0 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots = \sum_{i=1}^{\infty} \frac{3}{10^m}$$

SO WE CAN REASONABLY SAY THAT

$$\sum_{m=1}^{\infty} \frac{3}{10^m} = \frac{1}{3}$$

EXAMPLE: $\sum_{m=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \ldots$

THIS SUM GROWS WITHOUT BOUNDS SO WE CAN REASONABY SAY IT DIVERGES

EXAMPLE $\sum_{m=1}^{\infty} (-1)^m = -1 + 1 - 1 + 1 - 1 + 1 \ldots$

THIS SUM GOES BACK AND FORTH FOREVER BETWEEN -1 AND 0 SO WE CAN REASONABLY SAY IT DIVERGES

DEF:

THE N^{TH} PARTIAL SUM OF THE SERIES

$\sum_{i=1}^{\infty} a_m$ IS $S_N = \sum_{n=1}^{N} a_m = a_1 + \dots + a_N$.

THE PARTIAL SUMS FORM A SEQUENCE

$\{S_N\}_{N=1}^{\infty}$. IF S_N CONVERGES TO A

NUMBER S WE SAY $\underline{\sum_{m=1}^{\infty} a_m = S}$.

IF S_N DIVERGES WE SAY $\sum_{m=1}^{\infty} a_m$ DIVERGES.

EXAMPLE: (GEOMETRIC SERIES)

CONSIDER THE SUM

$$a + ar + ar^2 + ar^3 + ar^4 + \dots + ar^m + \dots = \sum_{m=0}^{\infty} ar^m$$

$\left(\text{OR } \sum_{m=1}^{\infty} ar^{m-1}\right)$ $\left(\text{EXAMPLE IN THE EXAMPLE: } \sum_{m=0}^{\infty} \frac{3}{10^m}\right)$

WE CAN WRITE DOWN THE PARTIAL SUM S_N

NICELY IF $r \neq 1$

$$S_N = a + ar + \dots + ar^N = \frac{a(r^N - 1)}{r-1} = \frac{a(1-r^N)}{1-r^N}$$

NOW NOTE: IF $r \geq 1$ THE SERIES

CLEARLY DIVERGES