THE DIVERGENCE TEST

SUPPOSE OUR SEQUENCE a_m DOES NOT CONVERGE TO 0. THIS IS LIKE SAYING THAT GIVEN SOME SMALL NUMBER d WE WILL ALWAYS FIND SOME a_m WITH $|a_m| > d$, NO MATTER HOW FAR WE GO.

NOW CONSIDER THE SERIES $\sum_{m=1}^{\infty} a_m$.

TO SAY IT CONVERGES TO A IS EQUIVALENT TO SAYING THAT FOR ANY SMALL NUMBER c EVENTUALLY WE'LL HAVE $|S_N - A| < c$ PERMANENTLY.

BUT! TAKE $c = \frac{d}{2}$, AND A N SUCH THAT $|a_{N+1}| > d$. THEN $|S_N - S_{N+1}| > d$ AND IT CANNOT HAPPEN THAT BOT $|S_N - A| < c$ AND $|S_{N+1} - A| < c$! WE MOVED TOO FAR! THIS SHOWS:

THM: FOR $\sum_{m=1}^{\infty} a_m$ TO CONVERGE IT IS NECESSARY THAT $a_m \to 0$.

CONEQUENTLY, IF $\lim_{n \to \infty} a_n \neq 0$ THEN $\sum_{n=1}^{\infty} a_n$ DIVERGES. (DIVERGENCE TEST)

EXAMPLE: $a_n = \frac{n}{n+1}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} 1 - \frac{1}{n} = 1$$ SO

$\sum_{n=1}^{\infty} a_n$ DIVERGES BY THE DIVERGENCE TEST.

WARNING: THE DIVERGENCE TEST ONLY TELLS US SOMETHING IF $a_n \not\to 0$. IT IS FULLY POSSIBLE THAT $a_n \to 0$ AND $\sum_{n=1}^{\infty} a_n$ STILL DIVERGES, AS IN

$\sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right)$. $\log\left(1 + \frac{1}{n}\right) \to 0$ BUT AS WE SAW THE SERIES DIVERGES.

EXAMPLE: $\sum_{n=1}^{\infty} \sin(n)$

THIS SERIES DOES NOT CONVERGE AS $\{a_n = \sin(n)\}$ DOES NOT CONVERGE TO 0; AN EASY WAY TO SEE IT IS THAT AS n WILL FALL ON $\left[\frac{1}{4}\pi + 2k\pi, \frac{3}{4}\pi + 2k\pi\right]$ INFINITELY MANY TIMES,

$\sin(m)$ WILL BE $\geq \frac{\sqrt{2}}{2}$ INFINITELY MANY TIMES.

SO, THE DIVERGENCE TEST WILL SHOW US THAT

$\sum_{m=1}^{\infty} a_m$ DIVERGES SOMETIMES, BUT IT'S A VERY ROUGH TEST... THERE ARE PLENTY OF SERIES THAT PASS IT BUT DIVERGE!

<u>EXAMPLE:</u> $\sum_{m=1}^{\infty} \frac{1}{m}$

LET'S REGROUP THE TERMS LIKE THIS

$$\underbrace{1}+\underbrace{\frac{1}{2}}+\underbrace{\frac{1}{3}+\frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}+\underbrace{\frac{1}{9}+\dots+\frac{1}{16}}$$

$$\underbrace{+\dots+\frac{1}{32}}+\underbrace{\dots+\dots+\frac{1}{64}}$$

THE TERMS FROM $m=2^i+1$ TO $m=2^{i+1}$ ARE ALL GREATER THAN THE LAST, $\frac{1}{2^{i+1}}$...

HOW MANY ARE THERE? 2^i! SO THEIR SUM IS $\geq \frac{1}{2}$! THEN

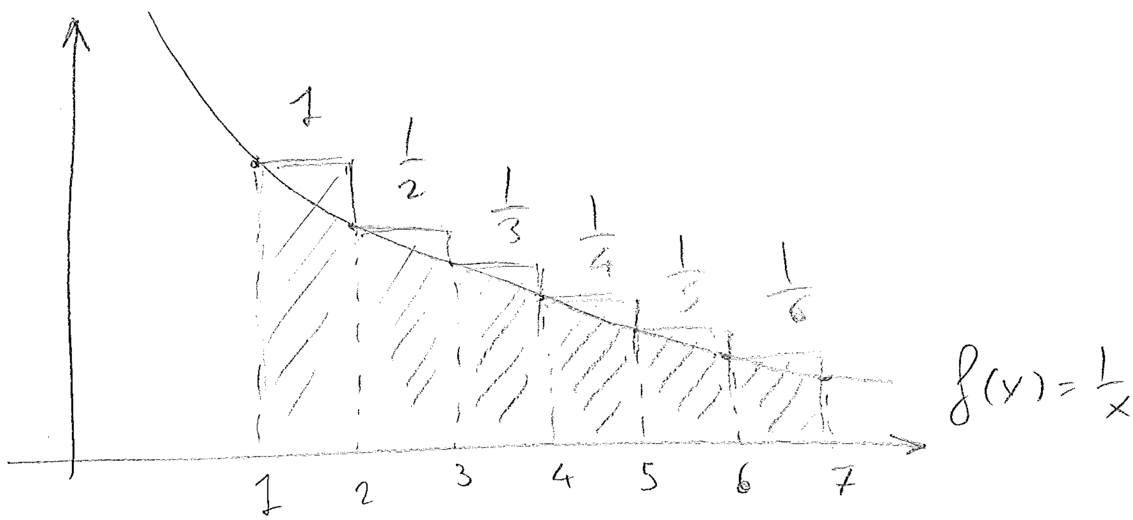
$\sum_{m=1}^{\infty} \frac{1}{m} \geq \sum_{i=1}^{\infty} \frac{1}{2} = \infty$! MORE PRECISELY:

$S_{2^N} \geq \frac{N+1}{2}$ SO $\lim_{N\to\infty} S_N$ DIVERGES.

THE INTEGRAL TEST

IS THERE A BETTER WAY TO SEE THAT $\sum_{m=1}^{\infty} \frac{1}{m}$ DIVERGES?

IDEA:

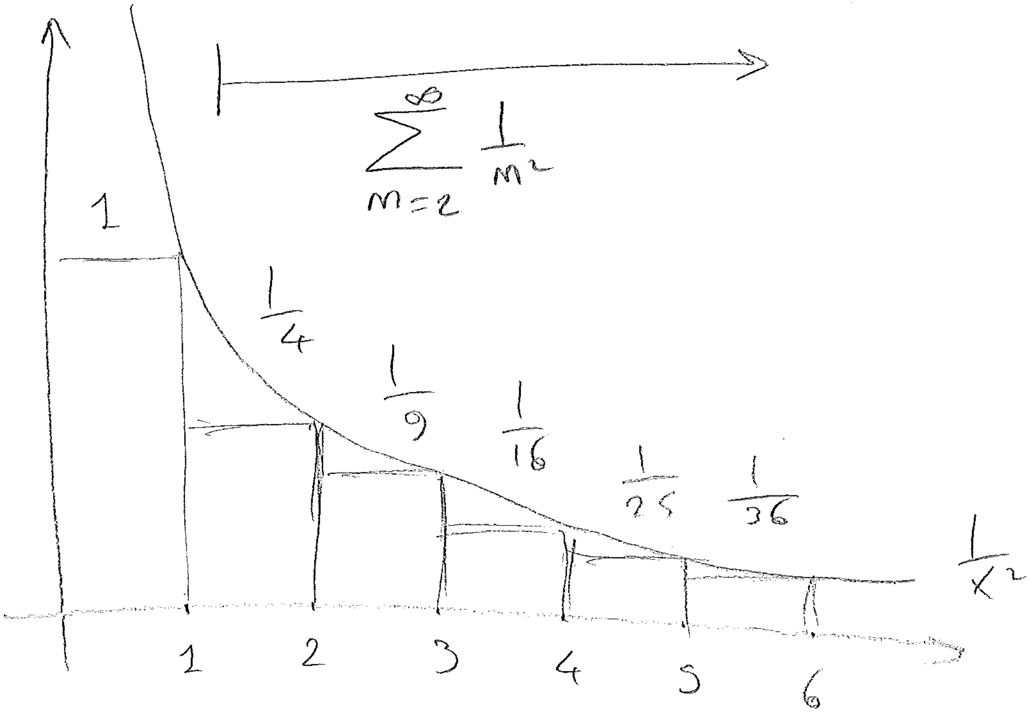


$$S_N \geq \int_1^N \frac{1}{x}\,dx = \log(N) \longrightarrow \infty \quad !!$$

WOW, THAT WAS EASY! BUT WHAT ABOUT SOMETHING THAT SHOULD CONVERGE, SUCH AS $\sum_{m=1}^{\infty} \frac{1}{m^2}$?

SMARTY-PANTS IDEA: COMPARE WITH TELESCOPING SERIES $\frac{1}{m(m+1)}$

MORE GENERAL IDEA: COMPARE WITH $\int_1^{\infty} \frac{1}{x^2}\,dx$



WE DON'T REALLY WANT TO LOOK AT

$\int_0^\infty \frac{1}{x^2}\,dx \ldots$ BUT WHAT IF WE REMOVE

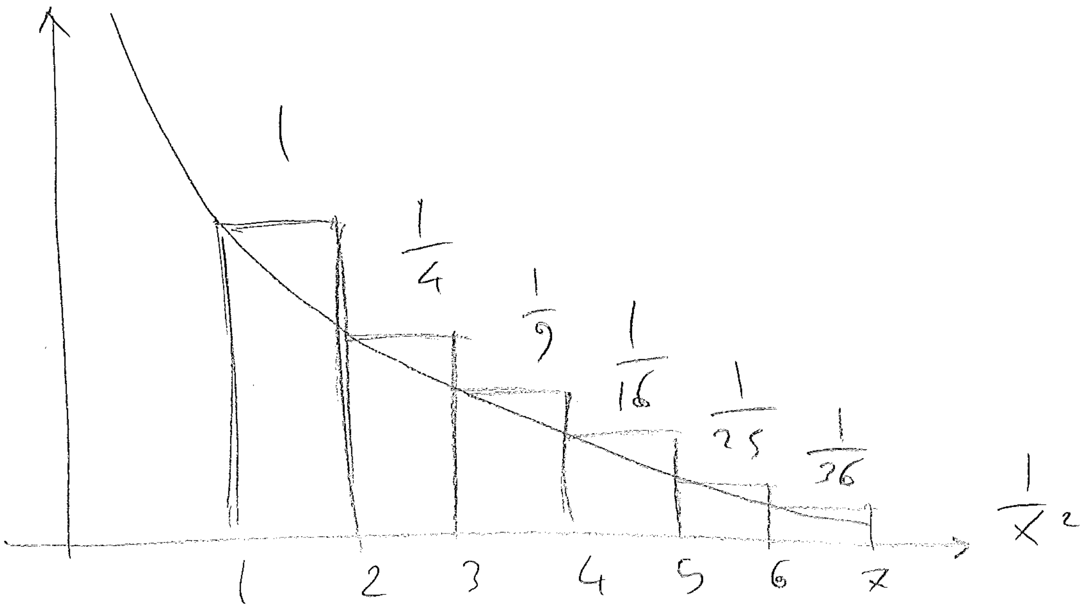
THE FIRST TERM? WE GET

$$S_N - 1 \le \int_1^N \frac{1}{x^2}\,dx \le \frac{1}{3} \;!! \quad \text{SO } S_N \le \frac{3}{2}$$

AS $\frac{1}{m^2}$ IS ALWAYS POSITIVE WE CAN CONCLUDE THAT S_N MUST CONVERGE TO SOME $A \le \frac{3}{2}$

<u>NOTE</u>: AN <u>INCREASING</u> SEQUENCE CAN EITHER <u>CONVERGE</u> OR <u>DIVERGE AT ∞</u>

NOW NOTE ONE MORE THING:



IF WE PICK A LEFT RULE SUM WE GET THAT

$$S_N \geq \int_1^N \frac{1}{x^2}\,dx \;!$$ SO KNOWING THAT S_N CONVERGES TELLS US THAT

$\int_1^\infty \frac{1}{x^2}\,dx$ CONVERGES! WE HAVE LAID DOWN ALL THE TOOLS FOR THE THEOREM:

<u>THM</u>: LET $f(x)$ A <u>POSITIVE</u> AND <u>DECREASING</u> FUNCTIONG, DEFINED ON $[1,\infty)$. THEN:

$$\int_1^\infty f(x)\,dx \text{ CONVERGES} \iff \sum_{n=1}^\infty f(n) \text{ CONVERGES}.$$

EXAMPLE $\displaystyle\sum_{m=1}^{\infty} \frac{1}{m^p}$

WE CAN USE THE INTEGRAL TEST WITH $f(x) = \frac{1}{x^p}$, WHICH IS BOTH POSITIVE AND DECREASING FOR $x>1$. THIS SHOWS THAT

$\displaystyle\sum_{m=1}^{\infty} \frac{1}{m^p}$ CONVERGES IF AND ONLY IF $p>1$

EXAMPLE $\displaystyle\sum_{m=1}^{\infty} \frac{1}{(m+1)\log(m+1)^p}$

AGAIN FOR $x>1$ $\frac{1}{(x+1)\log(x+1)^p}$ IS POSITIVE AND DECREASING, SO WE DO THE INTEGRAL TEST WITH

$$\int_1^{\infty} \frac{1}{(x+1)\log(x+1)^p}\,dx = \int_2^{\infty} \frac{1}{x\log(x)^p}\,dx \underset{u=\log(x)}{=}$$

$$\int_{u=\log(2)}^{\infty} \frac{1}{u^p}\,du$$ WHICH TELLS US THAT

$\displaystyle\sum_{m=1}^{\infty} \frac{1}{(m+1)\log(m+1)^p}$ CONVERGES IF AND ONLY IF $p>1$

EXAMPLE : $\displaystyle\sum_{m=1}^{\infty} \frac{\log(m)+10}{m^{\frac{3}{2}}}$

FIRST WE HAVE TO CHECK WETHER

$(\log(x)+10)\,x^{-\frac{3}{2}}$ IS DECREASING FOR $x > 1$

$$\left(\log(x)\cdot x^{-\frac{3}{2}} + 10x^{-\frac{3}{2}}\right)' = x^{-\frac{5}{2}} - \frac{3}{2}x^{-\frac{5}{2}}\log(x) - 15x^{-\frac{5}{2}}$$

$$= x^{-\frac{5}{2}}\left(-\log(x) - 14\right) < 0$$

$(\log(x)+10)\,x^{-\frac{3}{2}}$ IS CLEARLY POSITIVE, SO WE CAN USE THE INTEGRAL TEST.

$$\int (\log(x)+10)x^{-\frac{3}{2}}\,dx = -20x^{-\frac{1}{2}} + \int \log(x)\,x^{-\frac{3}{2}}\,dx =$$

$$\underset{\text{BY PARTS}}{=} -20x^{-\frac{1}{2}} - 2\log(x)\,x^{-\frac{1}{2}} + \int \frac{-2x^{-\frac{1}{2}}}{x}\,dx =$$

$$= -20x^{-\frac{1}{2}} - 2(\log(x)+2)x^{-\frac{1}{2}} = -2x^{-\frac{1}{2}}(12+2\log(x))$$

SO $\displaystyle\int_1^{\infty} (\log x + 10)\,x^{-\frac{3}{2}}\,dx = 24 + \lim_{R\to\infty} -2R^{-\frac{1}{2}}(12+2\log(R))$

$= 24$ THEN

BY THE INTEGRAL TEST $\displaystyle\sum_{m=1}^{\infty} \frac{\log(m)+10}{m^{\frac{3}{2}}}$

CONVERGES.

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)\log(\log(n+1))}$

WE'D LIKE TO USE THE INTEGRAL TEST, BUT $f(x) = \dfrac{1}{(x+1)\log(x+1)\log(\log(x+1))}$ IS NEGATIVE AT $x=1$ ...

IDEA: THE INTEGRAL TEST WORKS FINE EVEN IF WE START FARTHER! WE CAN COMPARE

$$\sum_{m=2}^{\infty} f(m) \quad \text{WITH} \quad \int_2^{\infty} f(x)\,dx, \qquad \text{AND}$$

FOR $x>2$ WE HAVE $x+1>3$ AND THE FUNCTIONS $(x+1)$, $\log(x+1)$, $\log(\log(x+1))$ ARE ALL POSITIVE $\left(\log(\log(3))>\log(1)=0\right)$ AND INCREASING SO $\dfrac{1}{(x+1)\log(x+1)\log(\log(x+1))}$ IS POSITIVE AND DECREASING.

NOW

$$\int_2^{\infty} \frac{1}{(x+1)\log(x+1)\log(\log(x+1))}\,dx = \int_3^{\infty} \frac{1}{x\log(x)\log(\log(x))}\,dx$$

$$\underset{u=\log(x)}{=} \int_{\log 3}^{\infty} \frac{1}{u\log u}\,du \underset{y=\log u}{=} \int_{\log(\log 3)}^{\infty} \frac{1}{y}\,dy \quad \text{WHICH}$$

DIVERGES. SO OUR SERIES DIVERGES.