WARM-UP

TRY TO COMPUTE

1) $\lim_{n \to \infty} 1 + \frac{\pi_n}{n}$ WHERE π_n = nth DIGIT OF π

2) $\lim_{n \to \infty} \arcsin\left(\frac{n+3}{2n+5}\right)$

SOLUTIONS IN THE NOTES

ANOTHER POWERFUL TOOL:

THM (SQUEEZE THEOREM):

IF $a_n \leq c_n \leq b_n$ FOR ALL (SUFFICIENTLY LARGE) n, AND

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

THEN $\lim_{n\to\infty} c_n = L$.

EXAMPLE: $\lim_{n\to\infty} 1 + \frac{\pi_n}{n}$

WHERE π_n = n-TH DECIMAL DIGIT OF π

BY DEFINITION $\pi_n \leq 9$ AND $\pi_n \geq 0$

SO

$$1 + \frac{\pi_n}{n} \leq 1 + \frac{9}{n} \quad \text{AND} \quad 1 + \frac{\pi_n}{n} \geq 1,$$

THEN

$$\underbrace{1}_{= a_n} \leq 1 + \frac{\pi_n}{n} \leq \underbrace{1 + \frac{9}{n}}_{= b_n}$$

NOW $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = 1$ SO

$$\lim_{n\to\infty} 1 + \frac{\pi_n}{n} = 1.$$

ONE LAST "ARITHMETHIC" PROPERTY

THM: SUPPOSE $\lim_{n \to \infty} a_n = L$, AND $g(x)$ CONTINUOUS AT $x = L$. THEN

$$\lim_{n \to \infty} g(a_n) = g(L).$$

<u>EXAMPLE</u>: $\lim_{n \to \infty} \arcsin\left(\frac{n+3}{2n+5}\right) = \lim_{n \to \infty} \arcsin(a_n)$

WHERE $a_n = \frac{n+3}{2n+5}$. WE KNOW THAT

$\lim_{n \to \infty} a_n = \frac{1}{2}$, SO $\lim_{n \to \infty} \arcsin\left(\frac{n+3}{2n+5}\right) =$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

SERIES

IF WE HAVE A SEQUENCE $\{a_n\}_{n=1}^{\infty}$ ANOTHER IMPORTANT OBJECT TO CONSIDER IS THE SERIES $\sum_{n=1}^{\infty} a_n$. ROUGHLY SPEAKING $\sum_{n=1}^{\infty} a_n$ IS THE SUM OF ALL THE NUMBERS a_n. WHAT DOES IT EVEN MEAN?

EXAMPLE: (DECIMAL EXPANSION)

TAKE $\frac{1}{3} = 0.3333\ldots$ THIS EXPANSION LITERALLY MEANS THAT

$$\frac{1}{3} = 0 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots = \sum_{i=1}^{\infty} \frac{3}{10^n}$$

SO WE CAN REASONABLY SAY THAT

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{1}{3}$$

EXAMPLE: $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \ldots$

THIS SUM GROWS WITHOUT BOUNDS SO WE CAN REASONABY SAY IT DIVERGES

EXAMPLE $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 \ldots$

THIS SUM GOES BACK AND FORTH FOREVER BETWEEN -1 AND 0 SO WE CAN REASONABLY SAY IT DIVERGES

DEF:

THE N^{TH} PARTIAL SUM OF THE SERIES

$\sum_{i=1}^{\infty} a_n$ IS $S_N = \sum_{n=1}^{N} a_m = a_1 + \ldots + a_N$.

THE PARTIAL SUMS FORM A SEQUENCE

$\{S_N\}_{N=1}^{\infty}$. IF S_N CONVERGES TO A

NUMBER S WE SAY $\underline{\sum_{m=1}^{\infty} a_m = S}$.

IF S_N DIVERGES WE SAY $\sum_{m=1}^{\infty} a_m$ DIVERGES.

EXAMPLE: (GEOMETRIC SERIES)

CONSIDER THE SUM

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^m + \ldots = \sum_{m=0}^{\infty} ar^m$$

$\left(\text{OR } \sum_{m=1}^{\infty} ar^{m-1}\right)$ (EXAMPLE IN THE EXAMPLE: $\sum_{m=0}^{\infty} \frac{3}{10^m}$)

WE CAN WRITE DOWN THE PARTIAL SUM S_N

NICELY IF $r \neq 1$

$$S_N = a + ar + \ldots + ar^N = \frac{a(r^N - 1)}{r - 1} = \frac{a(1 - r^N)}{1 - r^N}$$

NOW NOTE: IF $r \geq 1$ THE SERIES

CLEARLY DIVERGES

SO WE ARE LEFT WITH $r<1$. WE HAVE

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} a \cdot \frac{1-r^{\overset{\to 0}{N}}}{1-r} = \frac{a}{1-r}$$

EXAMPLE: (RECURRING DECIMAL)

CONSIDER A RECURRING DECIMAL, SUCH AS

$0.\overline{3542} = 0.354235423542\ldots$

WE CAN WRITE IT AS

$$0.\overline{3542} = \sum_{m=1}^{\infty} \frac{3542}{10.000^m} = \frac{3542}{10.000} + \frac{3542}{10.000^2} + \frac{3542}{10.000^3} + \ldots$$

NOW WE KNOW THAT

$$\sum_{m=0}^{\infty} 3542 \cdot \left(\frac{1}{10^4}\right)^m = 3542 \cdot \frac{1}{1-\frac{1}{10^4}} = 3542 \cdot \frac{10^4}{10^4-1}$$

SO $$0.\overline{3542} = \sum_{m=1}^{\infty} \frac{3542}{10^{4m}} = 3542 \cdot \frac{10^4}{10^4-1} - 3542$$

FIRST TERM $m=0$

$$= \frac{3542}{10^4-1} = \frac{3542}{9999}$$ A RATIONAL NUMBER!

NOTE: $$\sum_{m=1}^{\infty} \frac{a}{r^m} = \frac{a}{1-r} - a = a\left(-1 + \frac{1}{1-r}\right) = a\left(\frac{r}{1-r}\right)$$

IT'S OFTEN IMPOSSIBLE TO FIND S_N EXACTLY.

BEFORE GOING IN TO THE GENERAL CASE, LET'S LOOK AT ONE MORE

(RATHER ARTIFICIAL) "EASY" CASE

EXAMPLE (TELESCOPING SERIES)

$\sum_{m=1}^{\infty} \frac{1}{m(m+1)}$. WE CAN REWRITE

$\frac{1}{m(m+1)}$ AS $\frac{1}{m} - \frac{1}{m+1}$. THIS SHOWS

THAT

$$S_N = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{N(N+1)} =$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

ALL TERMS CANCEL OUT EXCEPT

1 AND $-\frac{1}{N+1}$ SO

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1 \quad \text{SO} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

EXAMPLE: (DIVERGENT TELESCOPING)

$$\sum_{m=1}^{\infty} \log\left(1+\frac{1}{m}\right) = \log(2) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots$$

$$S_N = \log(2) + \log\left(\frac{3}{2}\right) + \cdots + \log\left(\frac{N+1}{N}\right)$$

NOW $\log(n) + \log\left(\frac{n+1}{n}\right) =$

$\log\left(\frac{n+1}{n} \cdot n\right) = \log(n+1)$, SO APPLYING

TO $S_N = \sum_{n=1}^{N} \log\left(\frac{n+1}{n}\right)$ WE GET

$S_N = \log(N+1)$ WHICH DIVERGES!

SOME (OBVIOUS) ARITHMETIC RULES

<u>THM</u>: SAY $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, THEN

$$\sum_{n=1}^{\infty} a_n + b_n = A + B \qquad \sum_{n=1}^{\infty} a_n - b_n = A - B$$

$$\sum_{n=1}^{\infty} C a_n = CA$$

<u>EXAMPLE</u> $\sum_{n=1}^{\infty} \frac{1}{3^n} - \frac{5}{n(n+1)}$

WE KOW THAT $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{\frac{2}{3}} - 1 = \frac{1}{2}$

AND THAT $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ SO $\sum_{n=1}^{\infty} \frac{5}{n(n+1)} = 5 \cdot 1$

THEN $\sum_{n=1}^{\infty} \frac{1}{3^n} - \frac{5}{n(n+1)} = \frac{1}{2} - 5 = -\frac{9}{2}$