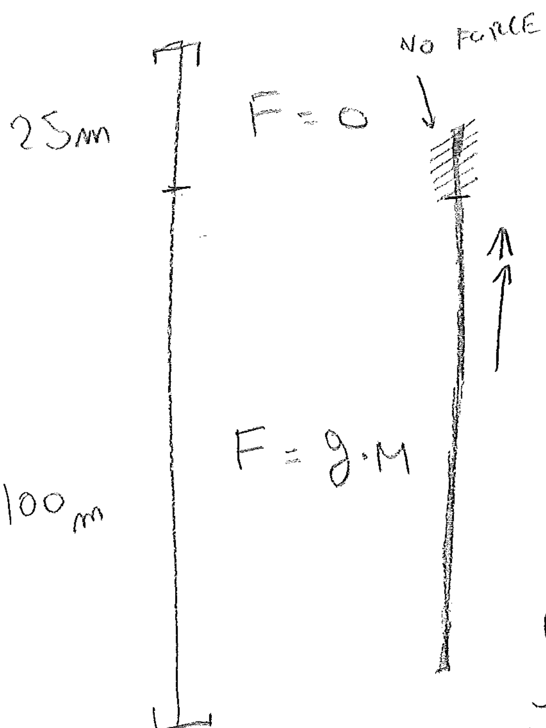
WARM-UP (ROPE PROBLEM, ALTERNATIVE SOLUTION)

A 100 m LONG ROPE, WEIGHTING 80 Kg, HANGS OVER THE EDGE OF A TALL BUILDING. WHAT'S THE WORK NEEDED TO PULL 25 m OF ROPE ON TOP OF THE BUILDING?

SOL: ON TOP OF THE BUILDING THE ROPE RESTS ON THE GROUND SO IT SUFFERS NO FORCE. WE CAN "STRAIGHTEN" IT AND SEE IT LIKE THIS:



SO THE ROPE MOVES 25 m AND THE TOTAL FORCE AFTER X METERS IS $g \cdot \frac{80}{100} \cdot (100 - x)$

Kg/m

REMAINING m OF ROPE

SO WE GET

$$\int_0^{25} g \cdot \frac{8}{10} \cdot (100 - x)\, dx = \frac{8g}{10}\left(2500 - \frac{625}{2}\right)$$

WARM-UP:

(1) A ROCKET OF MASS $M_R = 10^3$ Kg HAS $M_F = 10^4$ Kg OF FUEL. IT FLIES UP TO 10^6 m OF HEIGHT. THE FUEL IS CONSUMED AT A CONSTANT RATE (PROPORTIONAL TO HEIGHT) UNTIL IT IS 0 AT 10^6 m. HOW MUCH WORK WAS DONE BY THE REACTOR?

(2) HOW MUCH WORK IS NEEDED TO EMPTY A CONICAL (POINTING DOWN) TANK, $h = 10$ m, $r = 5$ m, HALF FULL OF WATER, FROM THE TOP?

SOL:

(1) MASS OF ROCKET AT HEIGHT

$$: \quad M_R + M_F\left(1 - \frac{x}{10^6}\right) = 10^3 + 10^4 - \frac{x}{100}$$

SO IF M_E = MASS OF EARTH, R = RADIUS OF EARTH,

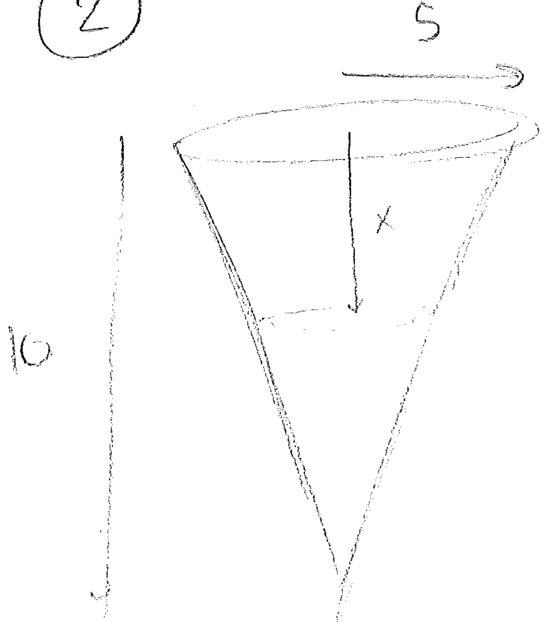
GRAV. CONSTANT

$$F(x) = \frac{\left(10^3 + 10^4 - \frac{x}{100}\right) \cdot M \cdot G}{(R+x)^2}$$

AND TOTAL WORK IS

$$W = \int_0^{10^6} \frac{\left(10^3 + 10^4 - \frac{x}{100}\right) \cdot M \cdot G}{(R+x)^2}\, dx$$

(2)



AREA OF SLICE AT DEPTH X

$$\left(\frac{(10-x)}{2}\right)^2 \cdot \pi$$

WATER HAS DENSITY 1

SO A Δ_x-WIDE STRIP WILL WEIGHT

$$\left(\frac{(10-x)}{2}\right)^2 \cdot 10^3 \pi\, \Delta_x$$

AND TO LIFT IT WE'LL NEED A WORK OF

$$\underset{\substack{\uparrow \\ \text{DISTANCE} \\ \text{MOVED}}}{x} \cdot \left(\underbrace{g\left(\frac{(10-x)}{2}\right)^2 \cdot 10^3 \pi \Delta_x}_{\text{FORCE}}\right)$$

SO TOTAL WORK IS

$$\int_5^{10} x \cdot \left(g\left(\frac{(10-x)}{2}\right)^2 \cdot 10^3 \pi\right) dx = \frac{10^3 \pi g}{4} \int_5^{10} x(10-x)^2\, dx$$

$$\approx 165 \cdot 10^3 \pi g (\cdot K_g)$$

AVERAGES

WE ALL KNOW WHAT THE AVERAGE OF A SET OF m NUMBERS IS:

$$\text{AVG}(\{y_1, \ldots, y_m\}) = \frac{y_1 + \cdots + y_m}{m}$$

BUT WHAT IF WE WANTED THE AVERAGE OF A FUNCTION $f(x)$? WHAT DOES IT EVEN MEAN?

IDEA: WHAT DO WE GET IF WE PICK m VALUES $f(x_1), \ldots, f(x_m)$ IN $[a,b]$ AND TAKE THE AVERAGE, FOR m VERY BIG?

$$\sum_{i=1}^{m} \frac{f(x_i)}{m} \quad \ldots \quad \text{DOES NOT RING A BELL.}$$

WHAT IF WE MULTIPLY BY $b-a$?

$$\sum_{i=1}^{n} f(x_i) \cdot \frac{b-a}{m} = \sum_{i=1}^{m} f(x_i) \Delta_x$$

THAT DOES RING A BELL! IF WE TAKE A SUFFICIENT NUMBER OF RANDOM POINTS, WE CAN THINK OF THEM AS

BEING EQUALLY DISTRIBUTED, SO "MORALLY"

$$\lim_{n\to\infty} \text{AVG}\{f(x_1), \ldots, f(x_n)\} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i)}{n}$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i)\Delta_x\right) \cdot \frac{1}{b-a} = \frac{1}{b-a} \int_a^b f(x)\,dx!$$

DEF: THE AVERAGE OF $f(x)$ ON $[a,b]$ IS $\frac{1}{b-a}\int_a^b f(x)\,dx$

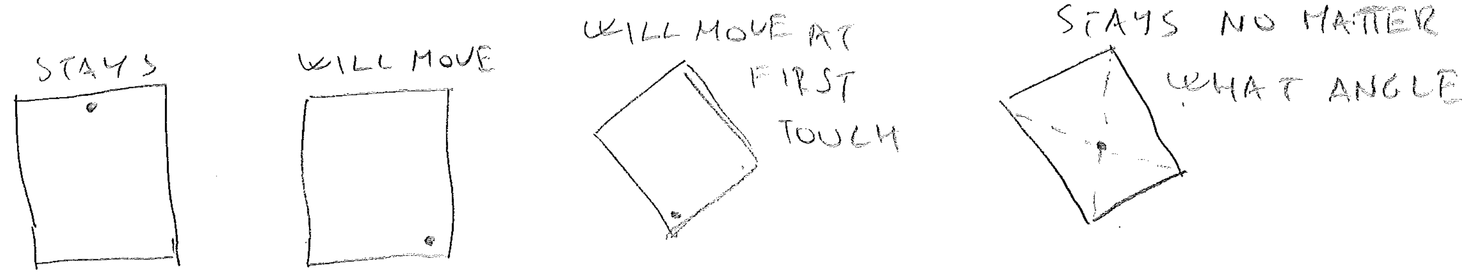
EXAMPLE: THE AVERAGE OF $\cos(x)$ ON $[0, \frac{\pi}{2}]$ IS

$$\frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \cos x\,dx = \frac{2}{\pi} \cdot \sin(x) \Big|_0^{\frac{\pi}{2}} = \frac{2}{\pi}$$

EXAMPLE: THE AVERAGE OF x^2 ON $[-2, 0]$ IS $\frac{1}{2}\int_{-2}^{0} x^2\,dx = \frac{x^3}{6}\Big|_{-2}^{0} = \frac{4}{3}$

CENTER OF MASS

EVER TRIED PINNING A POSTER TO A WALL?



THE LAST POINT IS CALLED THE <u>CENTER OF MASS</u>, AND IT'S A VERY IMPORTANT PHYSICAL OBJECT. FOR MANY PURPOSES (BUT NOT ALL BY ANY MEAN!) WE CAN REDUCE A BODY TO A POINT WITH THE SAME MASS LOCATED IN THE CENTER OF MASS.

IF WE HAVE A FINITE NUMBER OF POINTS IN POSITIONS (x_i, y_i) WITH MASSES m_i THEN THE <u>TOTAL MASS</u> M IS

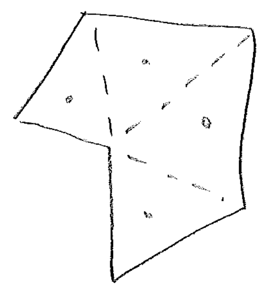
$\sum_{i=1}^{m} m_i$ AND THE CENTER OF MASS

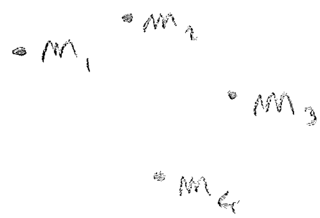
$(\hat{x}, \hat{y})$ IS GIVEN BY THE "WEIGHTED AVERAGES"

$$\hat{x} = \sum_{i=1}^{n} \frac{x_i \cdot m_i}{M}, \quad \hat{y} = \sum_{i=1}^{n} \frac{y_i \, m_i}{M}$$

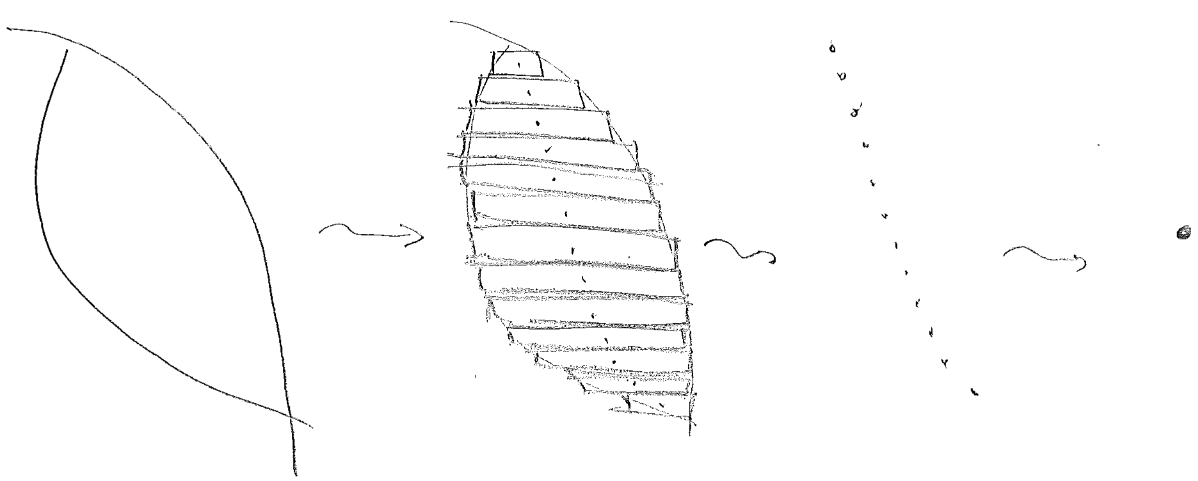
BUT WHAT IF WE HAVE A 2-d OBJECT, SUCH AS A METAL PLATE (of UNIFORM DENSITY ρ)?

IDEA: IF WE SUBDIVIDE AN OBJECT IN m PIECES, THE CENTER OF MASS (C.O.M.) OF THE OBJECT IS THE SAME AS THE C.O.M. OF m POINTS, EACH AT ITS RESPECTIVE PIECE'S C.O.M., EACH HAVING THE MASS OF THE RESPECTIVE PIECE.

SO C.O.M. OF  =

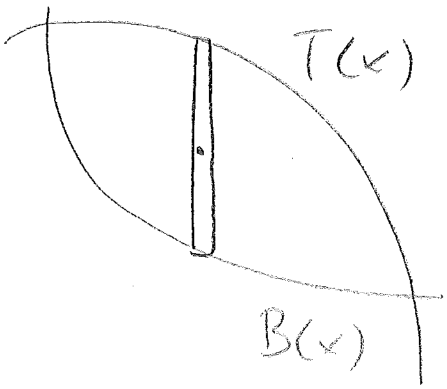
C.O.M. OF 

HOW DO WE USE THIS? WE <u>PANCAKE</u>!



SAY WE HAVE THE REGION

$$\{(x,y) \mid B(x) \leq y \leq T(x),\ a \leq x \leq b\}$$



HEIGHT OF A STRIPE:

$T(x)-B(x)$

WIDTH: Δ_x

C.O.M. OF THE STRIPE

IS AT IS AT $x, \frac{T(x)+B(x)}{2}$

MASS OF STRIPE IS

$\rho \cdot \Delta_x \cdot (T(x)-B(x))$ SO THE y-COORDINATE OF THE C.O.M. IN THE m-STRIPES APPROX IS

$$\frac{1}{M}\sum_{i=1}^{m}(T(x_i)-B(x_i))\cdot \rho \cdot \Delta_x \cdot \frac{T(x)+B(x)}{2} =$$

$$\frac{1}{M}\sum_{i=1}^{m}\frac{\rho(T(x_i)^2-B(x_i)^2)}{2}\cdot \Delta_x$$ WHERE x_i

ARE LEFT ENDP. OR RIGHT ENDP. OR MIDDLE...

GOING TO THE LIMIT $m \to \infty$

WE GET y-COORDINATE OF

C.O.M. = $\hat{y} = \frac{1}{2\rho A}\int_a^b \rho\left(T(x)^2 - B(x)^2\right)dx$

$= \frac{1}{2A}\int_a^b \left(T(x)^2 - B(x)^2\right)dx$ WHERE

A IS THE AREA OF OUR REGION

HOW ABOUT THE x-COORDINATE?

IT'S EASIER! DOING THE SAME

WE GET

x-COORD = $\hat{x} = \frac{1}{A}\int_a^b x\left(T(x) - B(x)\right)dx$

SO IN GENERAL:

DEF: THE C.O.M. $(\hat{x}, \hat{y})$ OF THE AREA BETWEEN $T(x)$ AND $B(x)$ RUNNING FROM a TO b IS

$$(\hat{x}, \hat{y}) = \left(\frac{1}{A}\int_a^b x\left|T(x) - B(x)\right|dx,\ \frac{1}{2A}\int_a^b \left|T(x)^2 - B(x)^2\right|dx\right)$$