WARM-UP

	EQUATION	PASSES THROUGH
1) SOLVE:	$\dfrac{dy}{dx} = x e^{x^2 - \log(y^2)}$,	$(0, 1)$

2) SOLVE: $x \dfrac{dy}{dx} = 2(y-4)$

SOL:

1) $\dfrac{dy}{dx} = x \dfrac{e^{x^2}}{y^2} \sim y^2 y' = x e^{x^2}$

So $\displaystyle\int y^2\,dy = \int x e^{x^2}\,dx \sim \frac{y^3}{3} = \frac{e^{x^2}}{2} + C$

$y = \sqrt[3]{\frac{3}{2} e^{x^2} + C}$. Now $y(0) = 1 = \sqrt[3]{\frac{3}{2} + C}$

So $C = -\frac{1}{2}$

2) $\dfrac{dy}{dx} = \dfrac{2(y-4)}{x} \sim \dfrac{y'}{y-4} = \dfrac{2}{x}$

$\sim \ldots \sim \log|y-4| = 2\log|x| + C$

$\sim |y-4| = |x|^2 \cdot e^C \sim y = Cx^2 + 4$

NOTE: C IS <u>ANY</u> REAL NUMBER. $y \equiv 4$ IS A SOLUTION.

EXAMPLE: $\dfrac{dy}{dx} - y\log(x) = \log\left(\frac{1}{x^2}\right)$

FIRST WE GET IT BACK TO STANDARD S.D.E. FORM

$$y' - y\log(x) = \log\left(\frac{1}{x^2}\right) \rightsquigarrow$$

$$y' = +y\log(x) + \log\left(\frac{1}{x^2}\right) \rightsquigarrow$$

$$y' = +y\log(x) - 2\log(x) \rightsquigarrow y' = \log(x)(y-2).$$

$$\underset{*\, y \neq 2}{\rightsquigarrow \dots \rightsquigarrow} \quad \int \frac{1}{y-2}\,dy = \int \log(x)\,dx$$

$$\log|y-2| = x\log x - x + C \quad \rightsquigarrow$$

$$|y-2| = e^{x\log x - x + C} \rightsquigarrow y - 2 = \overset{\pm e^{c}}{C}\, e^{x\log x - x}$$

LET'S FIND THE SOLUTION BASED ON $y(1)$ (IT HAS SOME PROBLEMS AT $x=0$)

$$y(1) = Ce^{-1} + 2 \rightsquigarrow C = e(y(1) - 2)$$

SEQUENCES

A SEQUENCE IS AN ORDERED INFINITE LIST OF NUMBERS. IT IS DENOTED

$$\{a_1, a_2, \dots, a_n, \dots\} \text{ OR } \{a_n\} \text{ OR } \{a_n\}_{n=1}^{\infty}$$

(NOTE: IT MAY ALSO START FROM A DIFFERENT NUMBER)

WHEN THERE IS AN EXPLICIT FUNCTION DESCRIBING THE SEQUENCE, WE'LL WRITE

$$\{a_n = f(n)\}_{n=1}^{\infty} \text{ OR SIMPLY } \{f(n)\}_{n=1}^{\infty}$$

EXAMPLE:

- $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$ OR $\left\{a_n = \frac{1}{n}\right\}_{n=1}^{\infty}$ OR $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$
- $\{1, -1, 1, \dots, (-1)^{n-1}, \dots\}$ OR $\{a_n = (-1)^{n-1}\}_{n=1}^{\infty}$ OR $\{(-1)^{n-1}\}_{n=1}^{\infty}$

NOTE THAT IN GENERAL THERE MAY BE NO REASONABLE FUNCTION TO DESCRIBE A SEQUENCE;

EXAMPLE:

$$\{a_n\} = \{3, 1, 4, 1, 5, 9, 2, 6, 5, \dots\}$$

$$\{a_n = n\text{-th DECIMAL DIGIT OF } \pi\}$$

A SEQUENCE $\{a_m\}$ <u>CONVERGES</u> TO A NUMBER A IF $\lim_{m \to \infty} a_m = A$

IT IS SAID TO <u>DIVERGE</u> IF $\lim_{m \to \infty} a_m$ DOES NOT EXIST OR IS $\pm\infty$.

EXAMPLE:

- $\left\{a_m = \frac{1}{m}\right\}$ CONVERGES TO $A = 0$ AS

$$\lim_{m \to \infty} \frac{1}{m} = 0$$

- $\left\{a_m = (-1)^{m-1}\right\}$ DIVERGES AS

$$\lim_{m \to \infty} (-1)^{m-1} \text{ DOES NOT EXIST}$$

- $\left\{a_m = m\right\}$ DIVERGES TO ∞ AS

$$\lim_{m \to \infty} m = \infty$$

- HOW ABOUT $\left\{\frac{m+3}{2m+5}\right\}_{m=1}^{\infty}$? WELL,

$$\frac{m+3}{2m+5} = \frac{m\left(1+\frac{3}{m}\right)}{m\cdot\left(2+\frac{5}{m}\right)} = \frac{1+\frac{3}{m}}{2+\frac{5}{m}} \quad \text{SO}$$

$$\lim_{n\to\infty} \frac{n+3}{2n+5} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{2+\frac{5}{n}} = \frac{1}{2}$$

THIS LOOKS LIKE WHAT WE WOULD DO FOR A REGULAR LIMIT... AND IN FACT:

THM: IF $\lim_{x\to\infty} f(x) = A$, THEN $\{a_n = f(n)\}$ CONVERGES TO A.

EXAMPLE: $\{a_n = e^{-n}\}$ THEN $\lim_{n\to\infty} a_n = \lim_{x\to\infty} e^{-x} = 0$

NOTE THAT THIS <u>DOES NOT</u> GO TWO WAYS!

EXAMPLE:

$f(x) = \sin(\pi x)$ $\quad \lim_{x\to\infty} \sin(\pi x)$ DNE AS $f(x)$ KEEPS OSCILLATING BETWEEN 1 AND -1.

BUT! $\{a_n = f(n)\}_{n=1}^{\infty} = \{a_n = \sin(\pi n)\}_{n=1}^{\infty}$

<u>DOES</u> CONVERGE AS $\sin(\pi n) = 0$ FOR ALL n!

LIMITS OF SEQUENCES BEHAVE AS WE GENERALLY EXPECT IN TERMS OF THE USUAL OPERATIONS:

THEOREM: IF $\{a_n\}$ CONVERGES TO A AND $\{b_n\}$ CONVERGES TO B THEN:

SEQUENCE	CONVERGES TO
$\{a_n + b_n\}$	$A + B$
$\{a_n - b_n\}$	$A - B$
$\{c a_n\}$	cA
$\{a_n \cdot b_n\}$	$A \cdot B$
* $\left\{\frac{a_n}{b_n}\right\}$	$\frac{A}{B}$

*: PROVIDED $b_n \neq 0$, $B \neq 0$

EXAMPLE: $\left\{-\frac{n+3}{2n+5} + 3e^{-n}\right\}$

$$\lim_{n\to\infty} -\frac{n+3}{2n+5} + 3e^{-n} = \lim_{n\to\infty} 3(\underbrace{e^{-n}}_{\text{CONV TO } 0}) - \underbrace{\left(\frac{n+3}{2n+5}\right)}_{\text{CONV TO } \frac{1}{2}} = 3\cdot 0 - \frac{1}{2}$$

$$= \frac{1}{2}$$

EXAMPLE : $\sum_{m=1}^{\infty} \frac{1}{m^p}$

AS $\int_1^{\infty} \frac{1}{x^p}\,dx$ CONVERGES IF AND ONLY IF $p>1$,

AND $\frac{1}{x^p}$ IS POSITIVE AND DECREASING,

BY THE INTEGRAL TEST $\sum_{m=1}^{\infty} \frac{1}{m^p}$ CONVERGES

IF AND ONLY IF $p>1$

EXAMPLE : $\sum_{m=2}^{\infty} \frac{1}{m\log(m)^p}$

WE WANT TO COMPARE WITH $\int_2^{\infty} \frac{1}{x\log(x)^p}\,dx$

- $x\log(x)>0$ FOR $x>2$ SO $\frac{1}{x\log(x)^p}>0$
- $\left(\frac{1}{x\log(x)^p}\right)' = \frac{-(\log(x)+1)}{(x\log(x)^p)^2} < 0$ SO $\frac{1}{x\log(x)}$
  IS DECREASING FOR
- $\int_2^{\infty} \frac{1}{x\log(x)^p}\,dx = \int_{\log(2)}^{\infty} \frac{1}{u^p}\,du$ CONVERGES IF AND ONLY

IF $p>1$ SO $\sum_{m=2}^{\infty} \frac{1}{m\log m}$ CONVERGES IF AND ONLY IF

$p>1$ BY THE INTEGRAL TEST.