WARM-UP / EXAMPLE

- COMPUTE $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$

- DOES $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$ CONVERGE?

SOL:

i) $$\int_{-\infty}^{\infty} e^{-|x|}\,dx \underset{\substack{\uparrow\\ \text{EVEN}\\ \text{FUNCTION}}}{=} 2\int_{0}^{\infty} e^{-|x|}\,dx = \lim_{R\to\infty} 2\int_{0}^{R} e^{-x}\,dx$$

$$= \lim_{R\to\infty} -2e^{-R} - 2(-1) = 2$$

ii) LET'S CONSIDER $\displaystyle\int_{1}^{\infty} e^{-x^2}\,dx$.

WE HAVE $e^{-x^2} \le e^{-|x|}$ FOR $x \ge 1$ SO

"TAMER THAN TAME" $\displaystyle\int_{1}^{\infty} e^{-x^2}\,dx$ CONVERGES.

THEN $$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \underbrace{2\int_{1}^{\infty} e^{-x^2}\,dx}_{\text{CONV}} + \underbrace{\int_{-1}^{1} e^{-x^2}\,dx}_{\text{ORDINARY}} \text{ CONVERGES.}$$

IDEA / DEFINITION (UNBOUNDED INTEGRAND)

i) IF $\int_{T}^{b} f(x)\,dx$ EXISTS FOR ALL $c < T < b$

THEN $$\int_{c}^{b} f(x)\,dx = \lim_{T \to c^+} \int_{T}^{b} f(x)\,dx$$

WHEN LIMIT EXISTS AND IS FINITE.

(SO WE'RE THINKING OF A VERTICAL ASYMPTOTE AT c)

ii) IF $\int_{a}^{T} f(x)\,dx$ EXISTS FOR ALL $a < T < b$

THEN $$\int_{a}^{c} f(x)\,dx = \lim_{T \to c^-} \int_{a}^{T} f(x)\,dx$$

WHEN LIMIT EXISTS AND IS FINITE

iii) IF BOT $\int_{a}^{c} f(x)\,dx$ AND $\int_{c}^{b} f(x)\,dx$

EXIST THEN $$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

THE INTEGRAL IS CONVERGENT IF IT EXISTS, DIVERGENT OTHERWISE

WHAT NOT TO DO

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = 2 \qquad \underline{\text{WRONG!}}$$

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx \text{ ; NEITHER}$$

EXIST SO $\int_{-1}^{1} \frac{1}{x^2}\,dx$ IS DIVERGENT

EXAMPLE:

$$\int_{0}^{1} \log(x)\,dx = \lim_{\tau \to 0^+} \int_{\tau}^{1} \log(x) = \lim_{\tau \to 0^+} -\tau\log(\tau) + \tau$$

$$+ 1\log(1) - 1 = -1 + \lim_{\tau \to 0^+} -\tau\log(\tau) + \tau = -1$$

EXAMPLE:

$$\int_{0}^{1} \frac{1}{x^p}\,dx = \lim_{\tau \to 0^+} \int_{\tau}^{1} \frac{1}{x^p}\,dx = \begin{cases} p \neq 1 & \dfrac{x^{1-p}}{1-p} \\ p = 1 & \log|x| \end{cases}$$

SO $p > 1$ $\quad \lim_{\tau \to 0^+} \frac{x^{1-p}}{1-p}\Big|_{\tau}^{1} = \frac{1}{1-p} - \lim_{\tau \to 0} \frac{\tau^{1-p}}{1-p} = \infty$ AS $1-p<0$

$p < 1$ $\quad \lim_{\tau \to 0^+} \frac{x^{1-p}}{1-p}\Big|_{\tau}^{1} = \frac{1}{1-p} - \lim_{\tau \to 0} \frac{\tau^{1-p}}{1-p} = \frac{1}{1-p}$ AS

$1-p>0$

$p = 1$ $\quad \lim_{\tau \to 0^+} \log|\tau|\Big|_{\tau}^{1} = \lim_{\tau \to 0^+} \log|\tau| = -\infty$

EXAMPLE: $\displaystyle\int_1^2 \frac{1}{x\log(x)^p}$

$$\int_T^2 \frac{1}{x\log(x)^p}\,dx \underset{u=\log x}{\uparrow}{=} \int_T^2 \frac{u'}{u^p}\,dx = \int_{u(T)}^{u(2)} \frac{1}{u^p}\,du$$

BUT $u(T) \to \log(1)=0$ FOR $T \to 1$

SO
$$\int_1^2 \frac{1}{x\log(x)^p}\,dx = \int_0^{\log(2)} \frac{1}{u^p}\,du = \begin{cases} \text{DNE} & p \geq 1 \\ -\dfrac{\log(2)^{1-p}}{p-1} & p<1 \end{cases}$$

REMARK: THERE IS NO p SUCH THAT

$\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$ EXISTS

EXAMPLE: DOES $\displaystyle\int_0^1 \frac{x}{x^{\frac{3}{2}}+x^2}\,dx$ CONVERGE?

FOR $x \leq 1$ $\quad \dfrac{x}{x^{\frac{3}{2}}+x^2} \leq \dfrac{x}{x^{\frac{3}{2}}} = \dfrac{1}{\sqrt{x}}$. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

CONV., SO "TAMER THAN TAME" TELLS US

THAT $\displaystyle\int_0^1 \frac{x}{x^{\frac{3}{2}}+x^2}\,dx$ CONVERGES.

MIXING AN MATCHING:

WE WANT, SAY $\int_{-\infty}^{\infty} \frac{1}{(x-2)x^2}\,dx$. DOES IT MAKE SENSE?

IF AN INTEGRAL IS "IMPROPER" IN MORE THAN ONE WAY, WE JUST BREAK IT DOWN UNTIL IT IS A SUM OF THE BASIC TYPES OF IMPROPER INTEGRALS, SO:

- $f(x)$ IS UNBOUNDED AT $x=2$, $x=0$
- WE HAVE $\pm\infty$

TO SOLVE THIS WE'LL PICK A $a<-2$, $-2<b<0$, $0<c$ AND WRITE IT AS

$$\int_{-\infty}^{a} \frac{1}{x^2(x-2)}\,dx + \int_{a}^{-2} \frac{1}{x^2(x-2)}\,dx + \int_{-2}^{b} \frac{1}{x^2(x-2)}\,dx +$$
$$\int_{b}^{0} \frac{1}{x^2(x-2)}\,dx + \int_{0}^{c} \frac{1}{x^2(x-2)}\,dx + \int_{c}^{\infty} \frac{1}{x^2(x-2)}\,dx$$

SEVERAL OF THESE ARE DIVERGENT, SO THE INTEGRAL IS DIVERGENT

WORK

WORK IS THE AMOUNT OF ENERGY IMPARTED ON A BODY BY A FORCE WHILE MOVING IT FROM a TO b. IT CAN BE SEEN AS THE AMOUNT OF ENERGY NEEDED TO MOVE AN OBJECT FROM a TO b AGAINST A FORCE (E.G. GRAVITY, A SPRING, A MAGNET)

IT'S DEFINED AS

$$W = \int_a^b F(x)\,dx$$

IN PARTICULAR, IF F IS CONSTANT, $W = F(b-a)$.

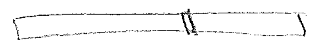
HOW DO WE COMPUTE IT?

WE PANCAKE OUR "PROBLEM" IN SLICES FOR WHICH THE FORCE IS CONSTANT, THEN INTEGRATE OVER THEM.

THE BEST WAY TO UNDERSTAND THIS IS BY (NUMEROUS) EXAMPLES.

For the following problems, we use the approximate value for acceleration due to gravity $g = -9.8m/s^2$ (or $g = 9.8m/s^2$, depending on your coordinate system).

(2) A cable hanging over the edge of a tall building is 40 meters long and weighs 60 kilograms. How much work is required to pull 10 meters of cable to the top of the building?

WE PANCAKE THE CABLE 

WEIGHT OF Δ_x OF CABLE $= \Delta_x \frac{60}{40}$

$= \frac{3}{2}\Delta_x$ SLICE AT HEIGHT X MOVES:

X IF $X<10$, 10 IF $X>10$ SO WE

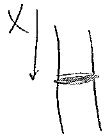
GET $\frac{3}{2}\int_0^{10} gx\,dx + \frac{3}{2}\int_{10}^{40} 10g\,dx = \frac{3}{2}g(50+300)$ m

$F = mg$ (MASS OF Δ_x CABLE $= \frac{3}{2}\Delta_x$)

(3) A 5-kilogram bucket containing 10 kilograms of water is lifted from the ground into the air by pulling in 20 meters of rope at a constant speed. The rope weighs 0.08 kilograms per meter. How much work was spent lifting the bucket and the rope?

BUCKET : $F = 15g$, DISTANCE $= 20m$

WORK $= 300\, g\cdot m$

ROPE : X  Δ_x m OF ROPE WEIGHT $0.08\, kg\,\Delta_x$

EACH PANCAKE MOVES X METRES

SO WORK $= \int_0^{20} \frac{g \cdot 8x}{100}\,dx = \frac{20^2}{2}\cdot\frac{8}{100}\cdot g \cdot m\cdot kg$

$= 16\, g\cdot m$

TOTAL $316\, g\cdot m\cdot kg \approx 3160 \frac{m^2}{s^2}\cdot kg$