WORK

WORK IS THE AMOUNT OF ENERGY IMPARTED ON A BODY BY A FORCE WHILE MOVING IT FROM a TO b. IT CAN BE SEEN AS THE AMOUNT OF ENERGY NEEDED TO MOVE AN OBJECT FROM a TO b AGAINST A FORCE (E.G. GRAVITY, A SPRING, A MAGNET)

IT'S DEFINED AS

$$W = \int_a^b F(x)\,dx$$

IN PARTICULAR, IF F IS CONSTANT, $W = F(b-a)$.

HOW DO WE COMPUTE IT?

WE PANCAKE OUR "PROBLEM" IN SLICES FOR WHICH THE FORCE IS CONSTANT, THEN INTEGRATE OVER THEM.

THE BEST WAY TO UNDERSTAND THIS IS BY (NUMEROUS) EXAMPLES.

§**2.1: Work** – In-class examples and additional problems

(1) According to Newtons' universal law of gravitation, the force between a planet of mass M and a probe of mass m is $F = \frac{GMm}{r^2}$, where r is the distance between them and $G \approx 6.67 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$ is the gravitational constant. Find the work required to launch a probe from the surface of a planet with radius R to a height of 1000 km. What if we want to launch the probe all the way to infinity?

i) $F(x) = \frac{GMm}{(R+x)^2}$ $\quad x = \text{HEIGHT}$

$10^6 \leftarrow$ METERS!

$$\text{WORK} = \int_0^{10^6} \frac{GMm}{(R+x)^2}\,dx = GMm \int_0^{10^6} \frac{1}{(R+x)^2}\,dx$$

$$\underset{u=R+x}{=} GMm \int_R^{10^6+R} \frac{1}{u^2}\,du = GMm \cdot \left(-\frac{1}{u}\right)\Bigg|_R^{10^6+R}$$

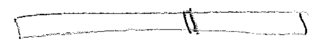
$$= GMm \left(\frac{1}{R} - \frac{1}{10^6+R}\right)$$

ii) WORK TO GET TO ∞ =

$$\lim_{h \to \infty} GMm\left(-\frac{1}{u}\right)\Bigg|_R^{R+h} = \frac{GMm}{R}$$

For the following problems, we use the approximate value for acceleration due to gravity $g = -9.8m/s^2$ (or $g = 9.8m/s^2$, depending on your coordinate system).

(2) A cable hanging over the edge of a tall building is 40 meters long and weighs 60 kilograms. How much work is required to pull 10 meters of cable to the top of the building?

WE PANCAKE THE CABLE 

WEIGHT OF Δ_x OF CABLE $= \Delta_x \frac{60}{40}$

$= \frac{3}{2}\Delta_x$ SLICE AT HEIGHT x MOVES:

x IF $x < 10$, 10 IF $x > 10$ SO WE

GET

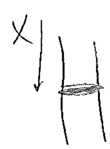
$$\frac{3}{2}\int_0^{10} gx\,dx + \frac{3}{2}\int_{10}^{40} 10g\,dx = \frac{3}{2}g\,(50+300)$$ m.kg

$F = mg$ (MASS OF Δ_x CABLE $= \frac{3}{2}\Delta_x$)

(3) A 5-kilogram bucket containing 10 kilograms of water is lifted from the ground into the air by pulling in 20 meters of rope at a constant speed. The rope weighs 0.08 kilograms per meter. How much work was spent lifting the bucket and the rope?

BUCKET: $F = 15g$, DISTANCE $= 20$ m

WORK $= 300\,g\cdot m$

ROPE:  Δ_x m OF ROPE WEIGHT $0.08\,kg\,\Delta_x$

EACH PANCAKE MOVES x METRES

SO WORK $= \int_0^{20} \frac{8x}{100}\,dx = \frac{20^2}{2}\cdot\frac{8}{100}\cdot g \cdot m\cdot kg$

$= 16\,g\cdot m\cdot kg$

TOTAL $316\,g\cdot m\cdot kg \simeq 3160\,\frac{m^2}{s^2}\cdot kg$

(4) Suppose, in the previous problem, that the bucket is leaking water at a constant rate. It finishes draining just as the bucket reaches the top. How much work was spent lifting the bucket and rope?

BUCKET LOSES WATER AT CONSTANT RATE, WATER $= 10\,\ell$ WHEN HEIGHT $= 0$, 0ℓ WHEN HEIGHT $= 20$ SO IT IS $10-\frac{x}{2}$

BUCKET MASS $= 5+\left(10-\frac{x}{2}\right)$ kg SO WORK

IS $\displaystyle\int_0^{20} g\left(5+\left(10-\frac{x}{2}\right)\right)dx = g\left(300-\frac{20^2}{4}\right) = 200\cdot g\,(m\cdot kg)$

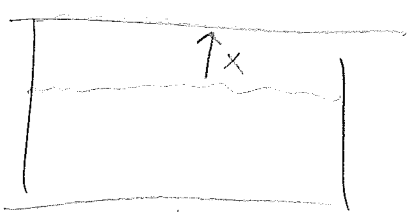
↑ HEIGHT

(5) A rectangular swimming pool measures 25 meters by 15 meters and is 9 meters deep. It is full of water, the density of which is 1000 kg/m^3. How much work is required to empty the pool by pumping the water over the side?

WE PANCAKE THE POOL.

WEIGHT OF A PANKAKE: $25\cdot 15\; 10^3\, \Delta_x$ kg

EACH PANKAKE MOVES BY X

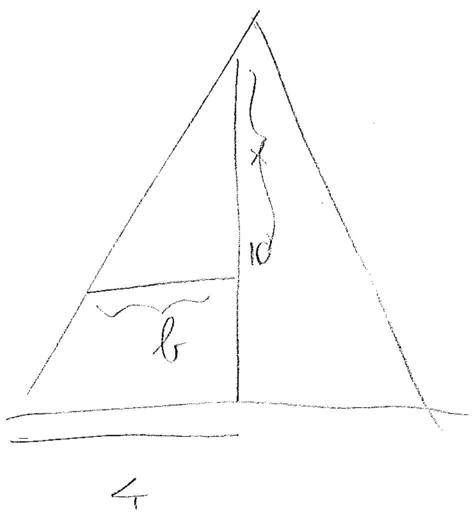


SO $\displaystyle\int_0^9 g\cdot x\cdot 25\cdot 15\cdot 10^3\,dx$

$= g\cdot 25\cdot 15\cdot 10^3\cdot \frac{81}{2}\,(kg\cdot m)$ IS THE WORK

(6) A conical tank measuring 10 meters high with base diameter 8 meters is full of a liquid which has density 810 kg/m^3. How much work does it take to pump the kerosene out of a spigot 1 m above the top of the tank?

PANCAKE THE TANK



$$\frac{10}{4} = \frac{x}{b} \qquad b = \frac{2x}{5}$$

AREA OF SLICE

$$\pi b^2 = \frac{4\pi}{25} x^2$$

WEIGHT OF Δ_x-SLICE $\simeq \frac{4\pi}{25} \cdot 810 x^2 \Delta_x$

HAS TO MOVE X METRES SO

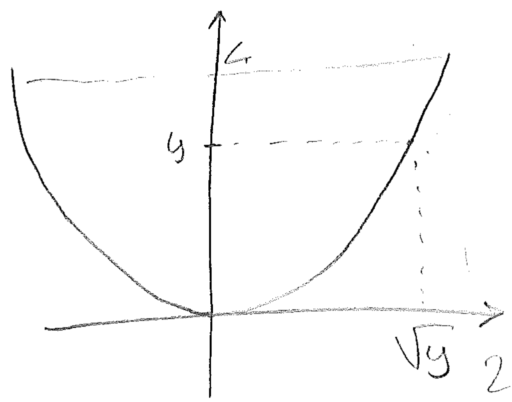
$$\frac{4\pi}{25} \cdot 810\, x^2 \cdot x = \frac{4\pi}{25} 810\, x^3$$

INTEGRATE

$$\int_0^{10} g \frac{4\pi}{25} \cdot 810\, x^3\, dx = g \frac{10^4}{4} \cdot \frac{4\pi}{25} \cdot 810$$

$$= g \cdot 10^2 \cdot 4\pi \cdot 810 \; (\cdot\, m \cdot kg)$$

(7) The graph of $y = x^2$ from $x = 0$ to $x = 2$ is revolved about the y-axis to form a tank that is then filled with salt water from the Dead Sea (which has density approximately 1200 kg/m^3). Assume that x and y-values are measured in meters. How much work does it take to pump all of the water to the top of the tank?



SLICE AREA:

$\pi(\sqrt{y})^2 = \pi \cdot y$

SLICE MASS:

$\pi \cdot y \cdot 1200 \, \Delta y$

HAS TO RISE $4-y$ m

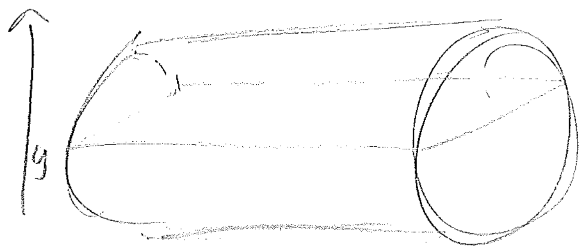
SO WORK ON SLICE $\pi \cdot y \cdot 1200 \cdot (4-y) \cdot g$

TOTAL WORK $\displaystyle\int_0^4 g \cdot \pi \cdot 1200 \cdot y(4-y)\, dy$

$$= \pi \cdot g \cdot 1200 \int_0^4 4y - y^2\, dy = \pi \cdot g \cdot 1200 \left(2y^2 - \frac{y^3}{3}\right)\Big|_0^4$$

$$= \pi \cdot g \cdot 1200 \cdot \left(32 - \frac{8}{3}\right) = \pi \cdot g \cdot 1200 \cdot \frac{88}{3} \, (\cdot \text{m} \cdot \text{kg})$$

(8) A right-circular cylindrical tank of height 10 meters and radius 5 meters is lying *horizontally* and is full of diesel fuel with density 900 kg/m^3. How much work is required to pump all of the fuel to a point 5 meters above the top of the tank?



$(y-5)^2 + x^2 = 25$

$x = \pm\sqrt{25-(y-5)^2}$ SLICE = RECTANGLE

OF AREA $2\cdot\sqrt{25-(y-5)^2} \times 10$

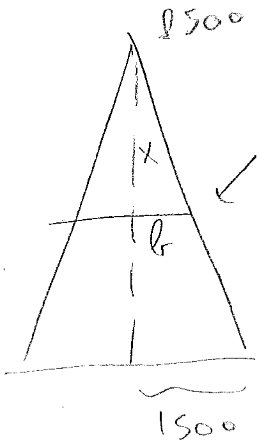
HAS TO MOVE $15-y$ METERS SO

WORK $= \left(2\cdot\sqrt{25-(y-5)^2}\right)\cdot 900\cdot(15-y)\cdot g$

TOTAL WORK $= g\int_0^{10}\left(2\cdot\sqrt{25-(y-5)^2}\cdot 900\cdot(15-y)\right)dy$

$= g\cdot 900\cdot 250\pi \;(\cdot m\cdot Kg)$

(9) WHAT'S THE WORK NEEDED TO THROW MOUNT EVEREST INTO SPACE? APPROXIMATE IT WITH A CONE WITH $h = 8500$ m, $r = 1500$ m, AND DENSITY 2.600 Kg/m^3. RADIUS OF EARTH $= R$, MASS OF EARTH $= M$



AREA: $\frac{x}{b} = \frac{8500}{1500} = \frac{17}{3}$

$b = \frac{3}{17} x \qquad \pi b^2 = \pi \frac{9}{17^2} x^2$

$\text{MASS} = \pi \frac{9}{17^2} x^2 \cdot 2600$

TO GO FROM $R + 8500 - x$ TO ∞

WORK IS $$\frac{GM\left(\frac{\pi 9 x^2}{17^2} \cdot 2600\right)}{R + 8500 - x}$$

TOTAL WORK $$\int_0^{8500} \frac{GM \pi \frac{9}{17^2} x^2 \cdot 2600}{R + 8500 - x}\, dx$$