<u>NUMERICAL INTEGRATION</u>

AS WE'VE SEEN, EVALUATING AN INTEGRAL CAN BE <u>HARD</u>, AND IT MIGHT BE THAT WE CANNOT DO IT AT ALL FOR SOME $f(x)$. SAME GOES FOR FINDING THE EXACT LIMIT OF A SEQUENCE OF RIEMANN SUMS.

SO WHAT CAN WE DO? OR MORE PRECISELY, WHAT CAN WE DO THAT WE CAN ALSO PROGRAM A COMPUTER TO DO?

<u>IDEA</u>: WE COULD PICK A RIEMANN SUM WITH n RECTANGLES, AND IF n IS BIG ENOUGH IT WILL BE VERY CLOSE TO THE ACTUAL INTEGRAL.

<u>MORE GENERAL IDEA</u>: WE CAN JUST DIVIDE OUR INTEGRAL IN n EQUAL PARTS AND IN EACH OF THESE USE SOME REASONABLE APPROXIMATION OF $f(x)$ THAT WE CAN INTEGRATE; THEN ADD UP THE PIECES.

<u>REMARK 1</u> PICKING A RIEMANN SUM IS JUST APPROXIMATING THE FUNCTION WITH A CONSTANT

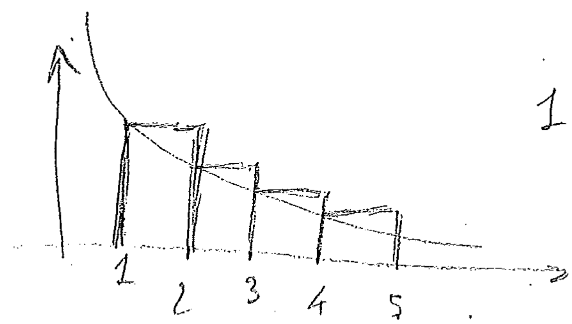
REMARK 2 AS IT HAPPENS IN GENERAL WITH APPROXIMATIONS (REMEMBER LINEAR AND QUADRATIC APPROX) WE WILL NEED AN **ERROR ESTIMATE**, OTHERWISE OUR APPROXIMATIONS WILL BE USELESS.

EXAMPLE:

WE WANT TO APPROXIMATE $\displaystyle\int_1^5 \frac{1}{x}\,dx = \log(5) \simeq 1.60943$

IN $n=4$ STEPS (INTERVALS)

LEFTPOINT RULE

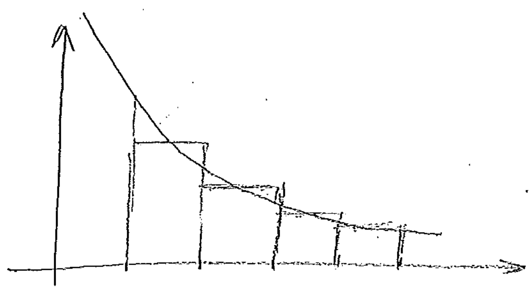


$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}$$

$$\simeq 2.08333$$

$|\text{ERROR}| \simeq 0.47$

MIDPOINT RULE

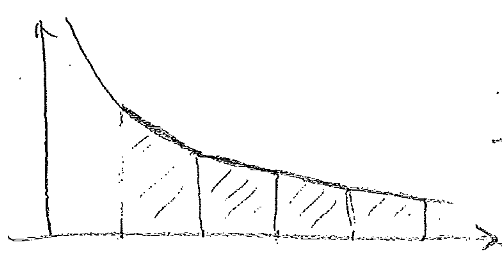


$$\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\frac{2}{9}=\frac{496}{315}$$

$$\simeq 1.57460$$

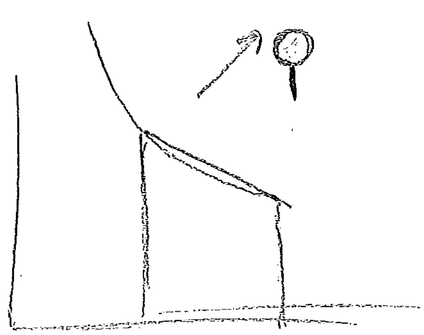
$|\text{ERROR}| \simeq 0.034$

TRAPEZOID RULE



$$\frac{1}{2}\left(\left(1+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}\right)\right)$$

$$=\frac{101}{60} \simeq 1.68333$$



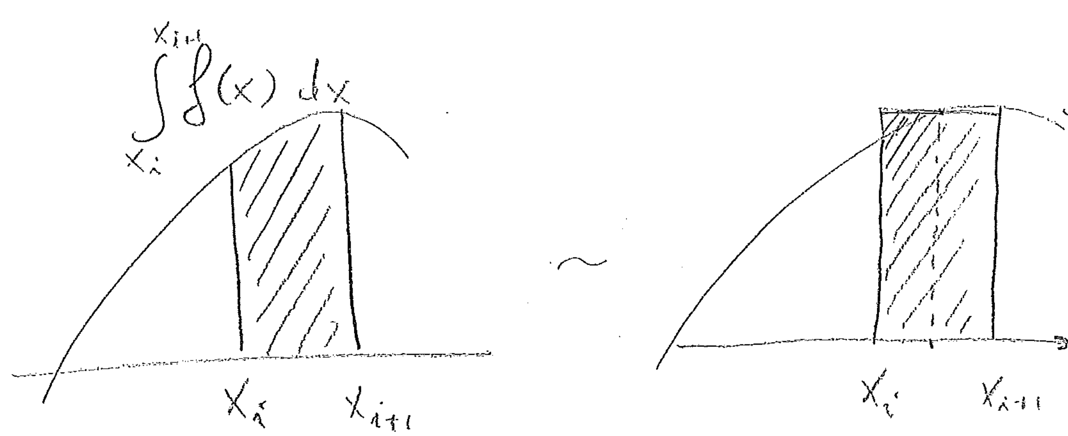
$|\text{ERROR}| \simeq 0.073$

WE'LL STUDY THREE TECHNIQUES OF NUMERICAL INTEGRATION:

MIDPOINT RULE, TRAPEZOIDAL RULE AND SIMPSON'S RULE (UNRELATED TO THE SIMPSONS, UNFORTUNATELY)

MIDPOINT RULE

WE APPROXIMATE THE INTEGRAL BY JUST TAKING A MIDPOINT RIEMANN SUM. THIS IS SURPRISINGLY EFFECTIVE!



$$\int_{x_i}^{x_{i+1}} f\left(\frac{x_i+x_{i+1}}{2}\right) dx = f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$$

SO WHAT WE ARE DOING IS APPROXIMATING

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \quad \text{WITH} \quad f\left(\frac{x_i+x_{i+1}}{2}\right)\cdot\left(x_{i+1}-x_i\right)$$

DEF: LET $\Delta_x = \frac{b-a}{n}$. THE MIDPOINT RULE APPROX (WITH n STEPS) IS:

$$\int_a^b f(x)\,dx \approx \left[f\left(a+\frac{\Delta_x}{2}\right)+f\left(a+\frac{3}{2}\Delta_x\right)+\dots+f\left(a+\left(n-\frac{1}{2}\right)\Delta_x\right)\right]\Delta_x$$

EX: $\int_0^1 \frac{4x}{1+x^2}\,dx$, APPROX WITH MID RULE, $M=4$

STEPS

- THE x VALUES: $a=0$, $b=1$, $\Delta_x = \frac{1}{4}$

SO THE x VALUES WILL BE

$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$

- $$\int_0^1 \frac{4x}{1+x^2}\,dx \approx \left[\frac{\frac{1}{2}}{1+\frac{1}{64}} + \frac{\frac{3}{2}}{1+\frac{9}{64}} + \frac{\frac{5}{2}}{1+\frac{25}{64}} + \frac{\frac{7}{2}}{1+\frac{49}{64}}\right]\cdot\frac{1}{4}$$

$= 1.396857\ldots$

LET'S SEE HOW GOOD OUR APPROX IS;

$$\int_0^1 \frac{4x}{1+x^2}\,dx = \log(4) = 1.386294\ldots$$

SO THE ERROR IS $|\log(4) - 1.396857\ldots| = 0.01056\ldots$

$\approx 10^{-2}$, OR $\approx 1\%$ OF THE ORIGINAL QTY.

DEF: SUPPOSE α IS AN APPROX OF A.

- ABSOLUTE ERROR $= |A-\alpha|$
- RELATIVE ERROR $= \frac{|A-\alpha|}{A}$
- % ERROR $= \frac{|A-\alpha|}{A}\cdot 100$

EX: $\displaystyle\int_{-\pi/2}^{\pi/2} \cos(x)\,dx$, 4 STEPS

- THE X-VALUES: $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$ $\Delta_x = \frac{\pi}{4}$

SO X VALUES

$-\frac{3}{8}\pi$, $-\frac{\pi}{8}$, $\frac{\pi}{8}$, $\frac{3}{8}\pi$

- $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx \approx \left[\cos\left(-\tfrac{3}{8}\pi\right) + \cos\left(-\tfrac{\pi}{8}\right) + \cos\left(\tfrac{\pi}{8}\right) + \cos\left(\tfrac{3}{8}\pi\right)\right]\cdot\frac{\pi}{4}$$

$= 2.05234\ldots$

EXACT VALUE IS $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx = 2$

SO ABSOLUTE ERROR $|2 - 2.05234\ldots| \approx 0.052$

RELATIVE ERROR $\frac{0.052\ldots}{2} \approx 0.026$

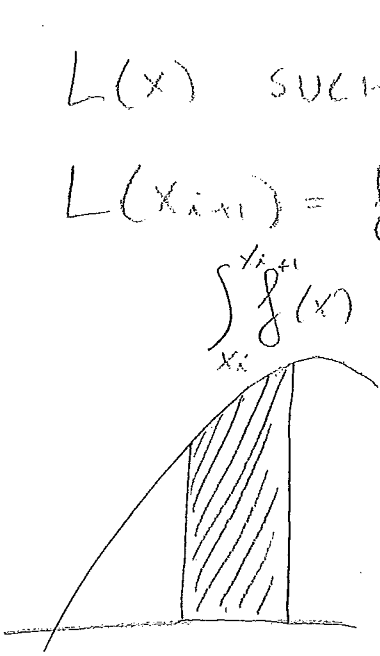
PERCENTAGE ERROR $\approx 2.6\%$

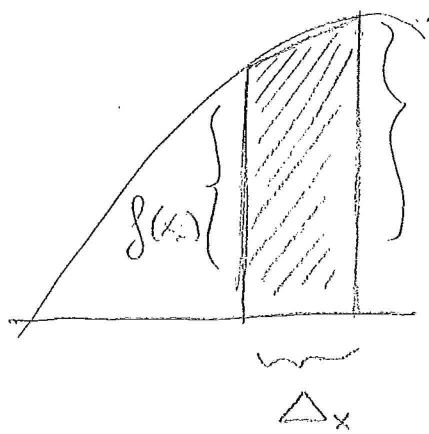
LATER WE'LL FIND <u>A PRIORI</u> BOUNDS FOR ALL THESE ERRORS

TRAPEZOIDAL RULE

WE APPROXIMATE THE FUNCTION BETWEEN x_i AND x_{i+1} WITH THE LINEAR FUNCTION $L(x)$ SUCH THAT $L(x_i) = f(x_i)$ AND $L(x_{i+1}) = f(x_{i+1})$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \qquad\qquad \int_{x_i}^{x_{i+1}} L(x)\,dx =$$





$$\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)\cdot \Delta_x$$

DEF: SET $\Delta_x = \frac{b-a}{m}$. THE TRAPEZOIDAL APPROXIMATION (WITH m STEPS) IS

$$\int_a^b f(x)\,dx \approx \left[\left(\frac{f(a)+f(a+\Delta_x)}{2}\right) + \left(\frac{f(a+\Delta_x)+f(a+2\Delta_x)}{2}\right) + \dots + \left(\frac{f(a+(m-1)\Delta_x)+f(b)}{2}\right)\right]\cdot \Delta_x = \left[f(a) + 2f(a+\Delta_x) + 2f(a+2\Delta_x) + \dots + 2f(a+(m-1)\Delta_x) + f(b)\right]\cdot \frac{\Delta_x}{2} =$$

$$\left[\frac{f(a)}{2} + \sum_{i=1}^{m-1} f(a+i\Delta_x) + \frac{f(b)}{2}\right]\cdot \Delta_x$$

EX: APPROXIMATE $\int_0^1 \frac{4x}{1+x^2}\,dx$ WITH THE TRAPEZOIDAL RULE, $m=4$

- WE HAVE $\Delta_x = \frac{1}{4}$, SO THE X-VALUES WILL BE $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

- THE APPROXIMATION IS

$$\int_0^1 \frac{4x}{1+x^2}\,dx \approx \left[\frac{0}{2} + \frac{1}{1+\frac{1}{16}} + \frac{2}{1+\frac{1}{4}} + \frac{3}{1+\frac{9}{16}} + \frac{4}{2}\cdot\frac{1}{2} \right]\cdot\frac{1}{4}$$

$$= 1.365294\ldots$$

OUR ERROR IS:

ABSOLUTE $|\log(4) - 1.365294\ldots| \doteq 0.02100\ldots$

RELATIVE $\dfrac{0.02100}{\log(4)} \approx 0.0151$

PERCENTAGE 1.5%

EX: APPROXIMATE $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx$ WITH TRAPEZOID RULE, $m=4$

- $\Delta x = \frac{\pi}{4}$, X VAL $-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}$

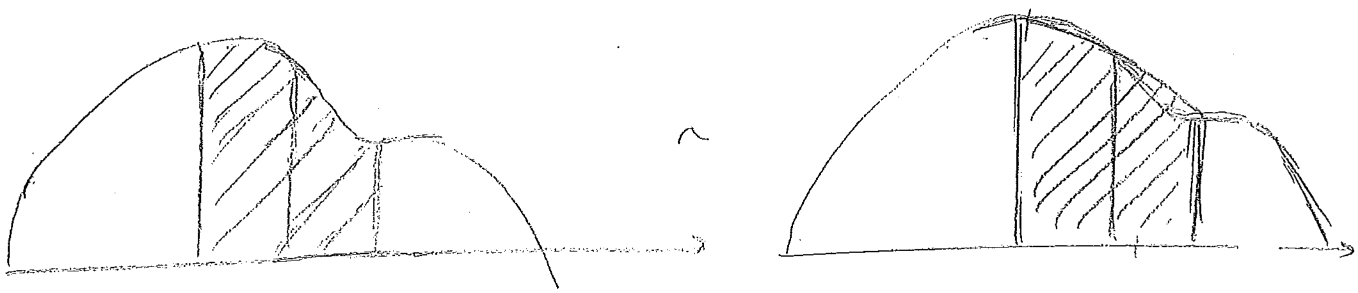
- $$\int_0^1 \cos(x)\,dx \approx \left[\frac{\cos(-\frac{\pi}{2})}{2} + \cos(-\tfrac{\pi}{4}) + \cos(0) + \cos(\tfrac{\pi}{4}) + \frac{\cos(\frac{\pi}{2})}{2} \right]\cdot\frac{\pi}{4} = (1+\sqrt{2})\cdot\frac{\pi}{4} \doteq 1.896118\ldots$$

ABSOLUTE ERROR $|2 - (1+\sqrt{2})\cdot\frac{\pi}{4}| = 0.103\ldots$

RELATIVE $\dfrac{0.103\ldots}{2} \approx 0.051$ PERCENTAGE 5.1%

SIMPSON'S RULE

ROUGHLY SPEAKING, SIMPSON'S RULE APPROXIMATES $f(x)$ ON TWO ADJACENT INTERVALS WITH A QUADRIC



DEF: SET $\Delta_x = \frac{b-a}{n}$, AND SUPPOSE n IS EVEN. THE SIMPSON'S RULE APPROX IS

$$\int_a^b f(x)\,dx \approx \left[f(a) + 4f(a+\Delta_x) + 2f(a+2\Delta_x) + \dots + 2f(a+(n-2)\Delta_x) + 4f(a+(n-1)\Delta_x) + f(b) \right] \frac{\Delta_x}{3}$$

$$= \left[f(a) + f(b) + 4 \sum_{\substack{i \text{ ODD} \\ 1 \le i \le n-1}} f(a+i\Delta_x) + 2 \sum_{\substack{j \text{ EVEN} \\ 2 \le i \le n-2}} f(a+j\Delta_x) \right] \frac{\Delta_x}{3}$$

EX: $\int_0^1 \frac{4x}{1+x^2}\,dx$ USING SIMPSON'S, $n=4$

- VALUES OF x: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$
- APPROX $\int_0^1 \frac{4x}{1+x^2}\,dx = \left[f(0) + f(1) + 4\left(f(\tfrac{1}{4}) + f(\tfrac{3}{4})\right) + 2f(\tfrac{1}{2}) \right] \frac{1}{12}$

$= \left[0 + 2 + 4\left(\frac{1}{\ } + \frac{3}{\ } \right) + 2\left(\frac{2}{\ } \right) \right] \cdot \frac{1}{\ } = \frac{1179}{\ } = 1.387058\ldots$

- ERROR:

ABSOLUTE $\left|\log(4) - 1.387058\ldots\right| = 0.00076\ldots$

RELATIVE $\dfrac{0.00076\ldots}{\log(4)} = 0.000551\ldots \approx 0.0005$

PERCENTAGE

$0.000551\ldots \cdot 100 \approx 0.05\,\%$

EX: $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx$ WITH SIMPSON'S, $n = 4$

- X VALUES: $-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{4}$

- APPROXIMATION $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx \approx$

$$\left[\cos\left(-\tfrac{\pi}{2}\right) + \cos\left(\tfrac{\pi}{2}\right) + 4\left(\cos\left(-\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\right) + 2\cos(0)\right]\frac{\pi}{12}$$

$$= \left[0 + 0 + 4\left(\sqrt{2}\right) + 2\right]\frac{\pi}{12} = \left(2\sqrt{2}+1\right)\frac{\pi}{6} \approx 2.00455\ldots$$

ERROR:

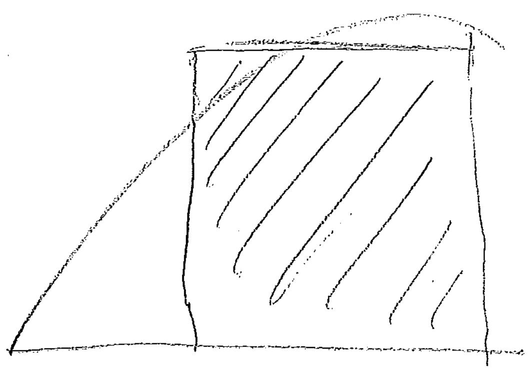
ABSOLUTE $\left|2 - \left(2\sqrt{2}+1\right)\frac{\pi}{6}\right| = 0.00455\ldots \approx 0.0045$

RELATIVE $\dfrac{0.00455\ldots}{2} = 0.00277\ldots \approx 0.0027$

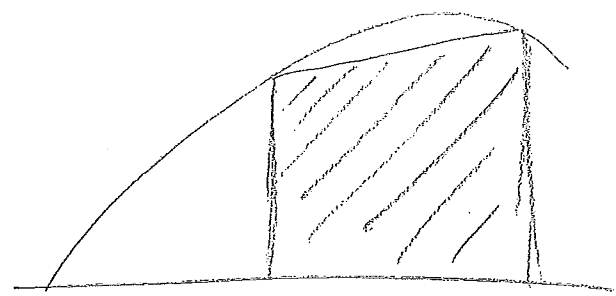
PERCENTAGE $\approx 0.0027 \times 100 = 0.27\,\%$

OPTIONAL

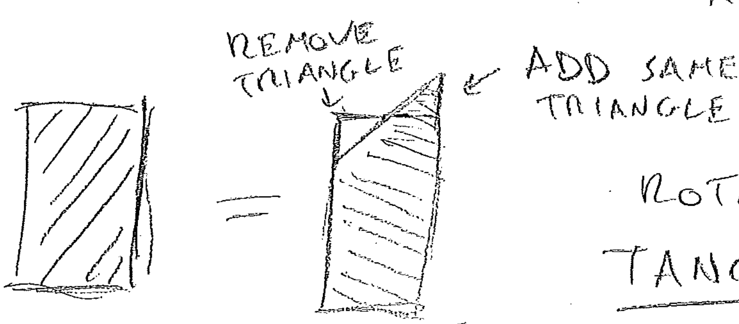
WHY DOES THE MIDPOINT RULE SEEM BETTER THAN TRAPEZOIDAL? IT SEEMS THAT THIS



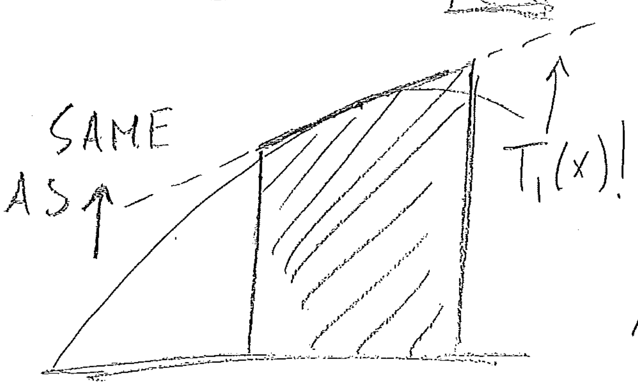
SHOULD BE LESS SMART THAN THIS!



IDEA: WITHOUT CHANGING THE AREA, WE CAN <u>ROTATE</u> THE TOP SEGMENT IN THE MIDPOINT RULE!



SO WE CAN ROTATE IT TO BE <u>TANGENT</u> TO $f(x)$!



IT'S THE <u>LINEAR APPROXIMATION</u> OF $f(x)$ AT THE MIDPOINT!!