ANSWER: (FROM PREV CLASS)

$$\frac{12x+30}{x(2x+3)(2x-5)} = \frac{A}{x} + \frac{B}{2x+3} + \frac{C}{2x-5}$$

LET'S TRY USING RESIDUES

$$\frac{12x+30}{(2x+3)(2x-5)} = A + \frac{xB}{2x+3} + \frac{xC}{2x-5}$$ AT $x=0$

$$\frac{30}{-15} = -2 = A$$

EVEN FASTER, WE DON'T NEED TO WRITE THE OTHER FACTORS

$$\left.\frac{12x+30}{x(2x-5)}\right|_{x=-\frac{3}{2}} = B \sim \frac{-18+30}{-\frac{3}{2}(-8)} = 1 = B$$

$$\left.\frac{12x+30}{x(2x+3)}\right|_{x=\frac{5}{2}} = 3 = C$$

So $$\frac{12x+30}{4x^3-4x^2-15x} = \frac{-2}{x} + \frac{1}{2x+3} + \frac{3}{2x-5}\ !$$

REMARK: WE PRETTY MUCH KNOW HOW TO DEAL WITH NON-REPEATED LINEAR FACTORS NOW.

IRREDUCIBLE QUADRATIC FACTORS

EXAMPLE:

REWRITE $\dfrac{8x-10}{4x^3-4x^2+5x}$ IN A FORM THAT WE CAN INTEGRATE.

WE HAVE $\dfrac{8x-10}{4x^3-4x^2+5x} = \dfrac{8x-10}{x(4x^2-4x+5)}$ IRREDUCIBLE

WE LOOK FOR A DECOMPOSITION AS

$$\frac{8x-10}{x(4x^2-4x+5)} = \frac{A+Bx}{4x^2-4x+5x} + \frac{C}{x}$$

- IDEA 1: MULTIPLY + ALGEBRA

$$8x-10 = Ax+Bx^2+C(4x^2-4x+5) \quad \text{SO}$$

$$\begin{cases} Bx^2+4Cx^2=0 \\ Ax-4Cx=8x \\ 5C=-10 \end{cases} \sim \begin{cases} B+4C=0 \\ A-4C=8 \\ C=-2 \end{cases} \sim \begin{cases} B=8 \\ A=0 \\ C=-2 \end{cases}$$

SO $$\frac{8x-10}{x(4x^2-4x+5)} = \frac{8x}{4x^2-4x+5} - \frac{2}{x}$$

NOTE: $\dfrac{8x}{4x^2-4x+5} = \dfrac{2x}{(x-\frac{1}{2})^2+1}$ FROM PREVIOUS EXAMPLE

IDEA 2: WE CAN MIX AND MATCH;

FIRST WE GET

$$\frac{8x-10}{4x^2-4x+5} = \frac{x(A+Bx)}{4x^2-4x+5x} + C \qquad \boxed{\text{EVAL AT } x=0}$$

$$-2 = \frac{-10}{5} = C$$

THEN WE DO THE ALGEBRA AS IN IDEA 1.

(USEFUL FOR MORE THAN 1 NON-REPEATED LINEAR FACTOR)

REPEATED LINEAR FACTORS

IF WE HAVE A LINEAR FACTOR APPEARING MORE THAN ONCE, SUCH AS

$$\frac{P(x)}{x^3(x+1)^2(x+2)}$$

WE'LL HAVE TO USE ONE CONSTANT PER POWER OF EACH LINEAR FACTOR, THAT IS:

$$\frac{P(x)}{x^3(x+1)^2(x+2)} = \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}\right) + \left(\frac{D}{x+1} + \frac{E}{(x+1)^2}\right) + \frac{F}{x+2}$$

EXAMPLE

$$\frac{x^2+1}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

TO FIND B, D

$$\frac{x^2+1}{(x+1)^2} = Ax^2 + Bx + C + \frac{Dx^3}{x+1} + \frac{Ex^3}{(x+1)^2} + \frac{Fx^3}{x+2}$$

EVAL AT $x=0$ $\quad 1 = B$

$$\frac{x^2+1}{x^2} = D + (x+1)(\dots)$$

EVAL AT $x=-1$ $\quad 2 = D$

NOW

$$x^2+1 = Ax(x+1)^2 + (x+1)^2 + Cx^2(x+1) + 2x^2$$

$$x^2+1 = Ax^3 + 2Ax^2 + Ax + x^2 + 2x + 1 + Cx^3 + Cx^2 + 2x^2$$

$$\begin{array}{l} \bullet \\ + \\ * \end{array}\left\{\begin{array}{l} 2x + Ax = 0 \\ 2Ax^2 + 2x^2 + x^2 + Cx^2 = x^2 \sim \\ Ax^3 + Cx^3 = 0 \end{array}\right. \quad \left\{\begin{array}{l} A + 2 = 0 \\ 2A + C + 2 = 0 \\ A + C = 0 \end{array}\right. \sim \left\{\begin{array}{l} A = -2 \\ C = 2 \end{array}\right.$$

So $$\frac{x^2+1}{(x+1)^2x^2} = \frac{-2}{x} + \frac{1}{x^2} + \frac{2}{x+1} + \frac{2}{(x+1)^2}$$

WE WON'T DEAL WITH REPEATED QUADRATIC FACTORS.

RECAP:

WE WANT TO INTEGRATE $\frac{N(x)}{D(x)}$

WHERE $\deg P(x) < \deg Q(x)$

(1) FACTORIZE $D(x) = \underbrace{L_1(x)^{a_1} \ldots L_r(x)^{a_r}}_{\text{LINEAR}} \underbrace{Q_1(x) \ldots Q_s(x)}_{\text{IRREDUCIBLE QUADRATIC}}$

(2) USE ANY COMBINATION OF THE IDEAS DESCRIBED TO WRITE $\frac{N(x)}{D(x)}$ AS

A SUM OF FUNCTIONS IN THE FORM

$$\frac{A}{L_i(x)^J} \quad \text{AND} \quad \frac{Ax+B}{Q_i(x)}$$

(3) INTEGRATE ONE BY ONE AND TAKE THE SUM.

WHAT IF $\deg N(x) > \deg D(x)$?

WE USE <u>LONG DIVISION</u>

LONG DIVISION (RECAP)

EXAMPLE: DIVIDE

$$16x^3 + 35 \quad \text{BY} \quad (2x+3)(x+1) = 2x^2 + 5x + 3$$

$16x^3 \qquad\qquad + 35$	$2x^2 + 5x + 3$
$16x^3 + 40x^2 + 24x$	$8x - 20$
$-40x^2 - 24x + 35$	
$-40x^2 - 100x - 60$	
$76x + 95$	

SO $$16x^3 + 35 = (2x^2 + 5x + 3)(8x - 20) + 76x + 95$$

THIS GIVES US:

- STEP 0: IF $\deg N(x) > \deg D(x)$

USE LONG DIV TO GET

$$N(x) = D(x)\,q(x) + R(x)$$

DEG < DEG D(x)

SO THAT $$\frac{N(x)}{D(x)} = q(x) + \frac{R(x)}{D(x)}$$

POLYNOMIAL

FINAL EXAMPLE:

INTEGRATE

$$\frac{16x^3+35}{2x^2+5x+3} = \frac{N(x)}{D(x)}$$

0 - $$\frac{16x^3+35}{2x^2+5x+3} = 8x-20 + \frac{76x+95}{(2x+3)(x+1)}$$

1 - $D(x) = (2x+3)(x+1)$

2 - $$\frac{76x+95}{(2x+3)(x+1)} = \frac{A}{2x+3} + \frac{B}{x+1}$$

$$\frac{76\left(-\frac{3}{2}\right)+95}{-\frac{3}{2}+1} = \frac{-19}{-\frac{1}{2}} = 38 = A$$

$$\frac{76(-1)+95}{(-2+3)} = 19 = B$$

3 - $$\int \frac{N(x)}{D(x)}dx = \int \frac{38}{2x+3}dx + \int \frac{19}{x+1}dx + \int 8x-20\,dx$$

$$= 19\log(2x+3) + 19\log(x+1) + 4x^2 - 20x + C$$