ERROR BOUNDS

WE NEED TO BE ABLE TO <u>PREDICT</u> THE <u>WORST CASE</u> ERROR.

<u>THM</u>:

ASSUME THAT $|f''(x)| \leq M$ FOR ALL x BETWEEN a AND b. THEN

- THE ABSOLUTE ERROR WHEN APPROXIMATING $\int_a^b f(x)\,dx$ WITH THE MID. RULE IS BOUNDED BY $\frac{M}{24}\frac{|b-a|^3}{n^2}$

- THE ABSOLUTE ERROR WHEN APPROXIMATING $\int_a^b f(x)\,dx$ WITH THE TRAPEZOIDAL RULE IS BOUNDED BY $\frac{M}{12}\frac{|b-a|^3}{n^2}$

NOW ASSUME $|f^{(4)}(x)| \leq L$ FOR ALL x BETWEEN a AND b. THEN

- THE ABSOLUTE ERROR WHEN APPROXIMATING $\int_a^b f(x)\,dx$ WITH THE SIMPSON'S RULE IS BOUNDED BY $\frac{L}{180}\frac{(b-a)^5}{n^4}$

REMARK: THESE SHOW THAT IF WE INCREASE THE NUMBER OF STEPS BY TEN TIMES, THE MIDPOINT AND TRAPEZOIDAL PRECISION INCREASES BY A FACTOR $10^2 = 100$, WHILE SIMPSON'S PRECISION INCREASES BY A FACTOR $10^4 = 10{,}000$!

EX: WHAT'S THE ACCURACY FOR THE MIDPOINT APPROX ON $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx$?

$$f(x) = \cos(x), \quad f''(x) = -\cos(x), \quad |f''(x)| \leq 1$$

SO WITH $M=1$, MAX ERROR

$$\frac{1}{24}\frac{\left|\frac{\pi}{2}+\frac{\pi}{2}\right|^3}{n^2} = \frac{\pi^3}{24}\cdot\frac{1}{n^2} \simeq \frac{1{,}29}{n^2}$$

EX: WHAT'S THE ACCURACY FOR THE TRAPEZOIDAL APPROX ON $\int_0^1 e^{-x^2}\,dx$?

$$\left(e^{-x^2}\right)'' = -2e^{-x^2} + 4x^2e^{-x^2} = 2(2x^2-1)e^{-x^2}$$

LET' BOUND EACH FACTOR

$4x^2-2$ IS INCREASING ON $[0,1]$, SO MAX/MIN AT ENDPOINTS $4\,0^2-2=-2$, $4\cdot1^2-2=2$

SO $|4x^2-2| \leq 2$ ON $[0,1]$

$e^{-x^2} \leq 1$ ON $[0,1]$ SO $\underline{M=2}$

THEN WE GET

$$\text{ABSOLUTE ERROR} \leq \frac{1}{6}\frac{|1-0|^3}{m^2} = \frac{1}{6m^2}$$

EX: HOW MANY STEPS DO WE NEED TO GET AN ACCURACY OF 10^{-6} IN PREVIOUS CASE?

WE WANT $\frac{1}{6m^2} \leq \frac{1}{10^6} \sim 10^6 \leq 6m^2$

$$m \geq \sqrt{\frac{10^6}{6}} = \frac{10^3}{\sqrt{6}} \approx 409$$

EX: WHAT IF WE USE MIDPOINT?

SAME BUT THE FORMULA IS

$$\frac{M}{24}\frac{|b-a|^3}{m^2} = \frac{1}{12}\cdot\frac{1}{m^2} \quad \text{SO} \quad m \geq \sqrt{\frac{10^6}{12}} = \frac{10^3}{2\sqrt{3}} \approx 289$$

EX: WHAT IF WE USE SIMPSONS?

$$\frac{d^4}{dx} e^{-x^2} = 4(4x^4 - 12x^2 + 3)e^{-x^2}$$

NOW $e^{-x^2} \leq 1$. LET'S SPLIT THE + AND − PARTS OF $4x^4 - 12x^2 + 3$

- $3 \leq 4x^4 + 3 \leq 7$ ON $[0,1]$.
- $-12 \leq 12x^2 \leq 0$ ON $[0,1]$.

SO $-9 \leq 4x^4 - 12x^2 + 3 \leq 7$

$M = 4 \cdot 9 \cdot 1 = 36$

THE ERROR IS THEN

$$\frac{36}{180}\frac{(1-0)^5}{m^4} = \frac{1}{5m^4}$$

SO $\quad 5m^4 \geq 10^6 \quad m^4 \geq 2\cdot 10^5 = 20.000$

$m \geq 22$.

IMPROPER INTEGRALS

IN PHISICS AND ENGINEERING WE OFTEN HAVE TO COMPUTE INTEGRALS ON INFINITE DOMAINS, OR LIMITS OF UNBOUNDED FUNCTIONS.

SOME EXAMPLES ARE

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \qquad \int_{-1}^1 \frac{1}{x^2}\,dx \qquad \int_0^1 \log(x)$$

UNBOUNDED FUNCTION

$$\int_0^\infty e^{-x}\,dx \qquad \int_1^\infty \frac{1}{\sqrt{x}}\,dx \qquad \int_0^\infty \frac{1}{1+x^2}\,dx$$

INFINITE DOMAIN

HOW DO WE DEAL WITH THESE?

DIVIDE AND CONQUER PLUS LIMITS, LIMITS

LIMITS.

IDEA / DEFINITION (INFINITE DOMAIN)

i) IF $\int_a^R f(x)\,dx$ EXIST FOR ALL $R > a$

THEN $$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$$

(EXISTS IF AND ONLY IF LIMIT EXISTS, AND IS FINITE)

ii) IF $\int_r^a f(x)\,dx$ EXISTS FOR ALL $r < a$

THEN $$\int_{-\infty}^a f(x)\,dx = \lim_{r\to-\infty} \int_r^R f(x)\,dx$$

(EXISTS IF AND ONLY IF LIMIT EXISTS AND IS FINITE)

iii) IF BOTH $\int_a^{+\infty} f(x)\,dx$ AND $\int_{-\infty}^a f(x)\,dx$

EXIST (ANY a CAN BE USED) THEN

$$\int_{-\infty}^\infty f(x)\,dx = \int_a^{+\infty} f(x)\,dx + \int_{-\infty}^a f(x)\,dx$$

IF THIS HAPPENS, THE INTEGRAL IS SAID TO BE CONVERGENT, DIVERGENT OTHERWISE.

EXAMPLE:

- $$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \arctan(R) - \arctan(1)$$

$$= \lim_{R\to\infty} \arctan(R) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

- $$\int_{-\infty}^{1} \frac{1}{x^2+1}\,dx = \lim_{r\to-\infty} \int_r^1 \frac{1}{x^2+1}\,dx = \lim_{r\to-\infty} +\frac{\pi}{4} + \arctan(r)$$

$$= \frac{3\pi}{4}$$ SO

- $$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$$

EXAMPLE: $\int_1^{\infty} \frac{1}{x^p}\,dx \qquad p>0$

$p>1$) $\int \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p} + C$ SO $\lim_{R\to\infty} \int_1^R \frac{1}{x^p}\,dx =$

$$\lim_{R\to\infty} \frac{R^{1-p}}{1-p} - \frac{1}{1-p} \underset{\substack{\uparrow \\ 1-p<0}}{=} -\frac{1}{1-p} = \boxed{\frac{1}{p-1} = \int_1^{\infty} \frac{1}{x^p}\,dx}$$

$p<1$) $\int \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p} + C$ SO $\lim \int_1^R \frac{1}{x^p}\,dx =$

$$\lim_{R\to\infty} \frac{R^{1-p}}{1-p} - \frac{1}{1-p} \underset{\substack{\uparrow \\ 1-p>0}}{=} \infty$$ DIVERGENT

$p=1$) $\lim \int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty} \log(|R|) = \infty$ DIVERG.

IDEA / DEFINITION (UNBOUNDED INTEGRAND)

i) IF $\int_{T}^{b} f(x)\,dx$ EXISTS FOR ALL $c < T < b$

THEN $$\int_{c}^{b} f(x)\,dx = \lim_{T \to c^+} \int_{T}^{b} f(x)\,dx$$

WHEN LIMIT EXISTS AND IS FINITE.

(SO WE'RE THINKING OF A VERTICAL ASYMPTOTE AT c)

ii) IF $\int_{a}^{T} f(x)\,dx$ EXISTS FOR ALL $a < T < b$

THEN $$\int_{a}^{c} f(x)\,dx = \lim_{T \to c^-} \int_{a}^{T} f(x)\,dx$$

WHEN LIMIT EXISTS AND IS FINITE

iii) IF BOT $\int_{a}^{c} f(x)\,dx$ AND $\int_{c}^{b} f(x)\,dx$

EXIST THEN $$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

THE INTEGRAL IS CONVERGENT IF IT EXISTS, DIVERGENT OTHERWISE

WHAT NOT TO DO

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = 2 \quad \underline{\text{WRONG!}}$$

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx \text{ ; NEITHER}$$

EXIST SO $\int_{-1}^{1} \frac{1}{x^2}\,dx$ IS DIVERGENT

EXAMPLE:

$$\int_0^1 \log(x)\,dx = \lim_{\tau \to 0^+} \int_\tau^1 \log(x) = \lim_{\tau \to 0^+} -\tau\log(\tau) + \tau$$

$$+ 1\log(1) - 1 = -1 + \lim_{\tau \to 0^+} -\tau\log(\tau) + \tau = -1$$

EXAMPLE:

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{\tau \to 0^+} \int_\tau^1 \frac{1}{x^p}\,dx = \begin{cases} p \neq 1 & \frac{x^{1-p}}{1-p} \\ p = 1 & \log|x| \end{cases}$$

SO $p > 1$ $\lim_{\tau \to 0^+} \frac{x^{1-p}}{1-p}\Big|_\tau^1 = \frac{1}{1-p} - \lim_{\tau \to 0} \frac{\tau^{1-p}}{1-p} = \infty$ AS $1-p<0$

$p < 1$ $\lim_{\tau \to 0^+} \frac{x^{1-p}}{1-p}\Big|_\tau^1 = \frac{1}{1-p} - \lim_{\tau \to 0} \frac{\tau^{1-p}}{1-p} = \frac{1}{1-p}$ AS $1-p > 0$

$p = 1$ $\lim_{\tau \to 0^+} \log|\tau|\Big|_\tau^1 = \lim_{\tau \to 0^+} \log|\tau| = -\infty$

MIXING AN MATCHING:

WE WANT, SAY $\int_{-\infty}^{\infty} \frac{1}{(x-2)x^2}\,dx$. DOES IT MAKE SENSE?

IF AN INTEGRAL IS "IMPROPER" IN MORE THAN ONE WAY, WE JUST BREAK IT DOWN UNTIL IT IS A SUM OF THE BASIC TYPES OF IMPROPER INTEGRALS, SO:

- $f(x)$ IS UNBOUNDED AT $x=2$, $x=0$
- WE HAVE $\pm\infty$

TO SOLVE THIS WE'LL PICK A $a<-2$, $-2<b<0$, $0<c$ AND WRITE IT AS

$$\int_{-\infty}^{a} \frac{1}{x^2(x-2)}\,dx + \int_{a}^{-2} \frac{1}{x^2(x-2)}\,dx + \int_{-2}^{b} \frac{1}{x^2(x-2)}\,dx +$$

$$\int_{b}^{0} \frac{1}{x^2(x-2)}\,dx + \int_{0}^{c} \frac{1}{x^2(x-2)}\,dx + \int_{c}^{\infty} \frac{1}{x^2(x-2)}\,dx$$

SEVERAL OF THESE ARE DIVERGENT, SO THE INTEGRAL IS DIVERGENT