WE WILL USE THE FOLLOWING THREE SUBSTITUTIONS:

$\sqrt{a^2 - x^2}$: $x = a\sin(u)$

$\sqrt{a^2 + x^2}$: $x = a\tan(u)$

$\sqrt{x^2 - a^2}$: $x = a\sec(u)$

So THAT

$$\sqrt{a^2 - x^2} \rightsquigarrow \sqrt{a^2 - a^2\sin(u)^2} = \sqrt{a^2\cos(u)^2} = |a\cos(u)|$$

$$\sqrt{a^2 + x^2} \rightsquigarrow \sqrt{a^2 + a^2\tan(u)^2} = \sqrt{a^2\sec(u)^2} = |a\sec(u)|$$

$$\sqrt{x^2 - a^2} \rightsquigarrow \sqrt{a^2 - a^2\sec(u)^2} = \sqrt{a^2\tan(u)^2} = |a\tan(u)|$$

EXAMPLE :

$\sin(u) = x$ $\qquad$ $\frac{dx}{du} = \cos(u)$

$$\int \frac{x^2}{\sqrt{1 - x^2}}\, dx = \int \frac{\sin(x)^2}{\sqrt{1 - \sin(u)^2}} \cos(u)\, du =$$

$$\int \frac{\sin(u)^2}{\sqrt{\cos(u)^2}} \cos(u)\, du = \int \sin(u)^2\, du =$$

POSITIVE ON $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\qquad$ $\frac{1}{2}\int 1 - \cos(2u)\, du =$

$$= \frac{u}{2} - \frac{1}{4}\sin(2u) + C$$

; NOW $\quad x = \sin(u)$, $\; u = \arcsin(x)$

$$= \frac{\arcsin(x)}{2} - \frac{1}{4}\sin(2\arcsin(x)) + C$$

IT'S NOT OVER:

$$\frac{\arcsin(x)}{2} - \frac{1}{4}\sin(2\arcsin(x)) + C =$$

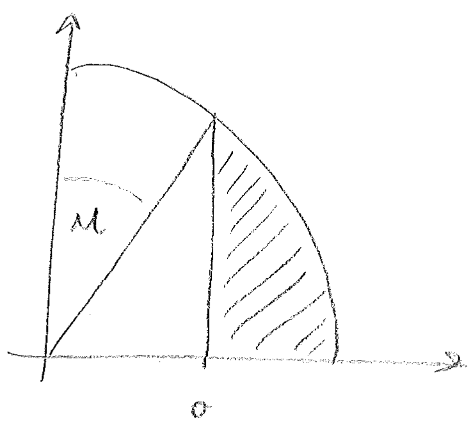
$$\frac{\arcsin(x)}{2} - \frac{1}{4}\cdot 2\sin(\arcsin(x))\cos(\arcsin(x)) + C$$

$$= \frac{\arcsin(x)}{2} - \frac{1}{2}\cdot x\cdot\sqrt{1-x^2} + C$$

EXAMPLE:

$$\int_a^r \sqrt{r^2-x^2}\,dx \qquad x = r\sin(u)$$

AND $u = \arcsin\frac{x}{r}$



EXTREMES OF INTEGRATION:

$u = \arcsin\frac{a}{r}$

$u = \arcsin\frac{r}{r} = \arcsin(1) = \frac{\pi}{2}$

So
$$\int_a^r \sqrt{r^2-x^2}\,dx = \int_{\arcsin\frac{a}{r}}^{\frac{\pi}{2}} \sqrt{r^2-r^2\sin(u)^2}\cdot r\cos(u)\,du$$

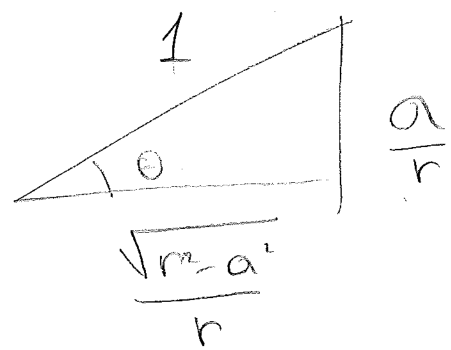
$$= \int_{\arcsin\frac{a}{r}}^{\frac{\pi}{2}} r^2\sqrt{1-\sin(u)^2}\cdot\cos(u)\,du = r^2\int_{\arcsin\frac{a}{r}}^{\frac{\pi}{2}}\cos(u)^2\,du$$

$$= r^2\int\frac{1+\cos(2u)}{2}\,du = \frac{r^2}{2}\left(u+\frac{\sin(2u)}{2}\right)\Bigg|_{\arcsin\left(\frac{a}{r}\right)}^{\frac{\pi}{2}}$$

$$= \frac{r^2}{2}\left(\frac{\pi}{2} + \frac{\sin(\pi)}{2} - \arcsin(a/r) - \frac{\sin(2\arcsin(a/r))}{2}\right)$$

$$= \frac{r^2}{2}\left(\frac{\pi}{2} - \arcsin(a/r) - \frac{2\sin(\arcsin(a/r))\cos(\arcsin(a/r))}{2}\right)$$

$$= \frac{r^2}{2}\left(\frac{\pi}{2} - \arcsin(a/r)\right) - \frac{a}{r}\sqrt{1-\frac{a^2}{r^2}}$$



EXAMPLE: $\int \frac{1}{x^2\sqrt{9+x^2}}\,dx$

$\sqrt{9+x^2} = \sqrt{3^2+x^2}$ SO WE PICK

$x = 3\tan u \quad \left(u = \arctan\left(\frac{x}{3}\right)\right)$

SO $\sqrt{9+x^2} = \sqrt{9+9\tan(u)^2} = 3\sqrt{\sec(u)^2}$

$$\int \frac{1}{x^2\sqrt{9+x^2}}\,dx = \int \frac{1}{27\tan(u)^2\,|\sec(u)|}\cdot 3\sec(u)^2\,du \Big|_{u=u(x)}$$

$$= \frac{1}{9}\int \frac{\sec(u)}{\tan(u)^2}\,du \Big|_{u=u(x)} = \frac{1}{9}\int \frac{\cos(u)}{\sin(u)^2}\,du \Big|_{u=u(x)}$$

NOW WE CAN WRITE $y = \sin u$

$$\frac{1}{9}\int \frac{\cos(u)}{\sin(u)^2}\,du = \frac{1}{9}\int \frac{y'}{y^2}\,du = \frac{1}{9}\int y^{-2}\,dy\Big|_{y=y(u)}$$

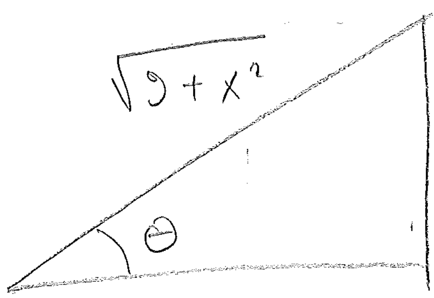
$$= -\frac{1}{9y} + C\Big|_{y=y(u)} = -\frac{1}{9\sin(u)} + C$$

NOW WE HAVE TO RECALL THAT

$u = \arctan\left(\frac{x}{3}\right)$

$$\int \frac{1}{x^2\sqrt{9+x^2}}\,dx = -\frac{1}{9}\,\frac{1}{\sin\left(\arctan\left(\frac{x}{3}\right)\right)} + C$$

WHAT'S $\sin\left(\arctan\left(\frac{x}{3}\right)\right)$?



$\sin(\theta) =$

$$\frac{x}{3}\cdot\frac{1}{\sqrt{\frac{x^2}{9}+1}} =$$

$$= \frac{x}{\sqrt{x^2+9}}$$

$$\int \frac{1}{x^2\sqrt{9+x^2}}\,dx = -\frac{1}{9}\,\frac{\sqrt{x^2+9}}{x} + C$$

EXAMPLE: $\int \frac{x^3}{\sqrt{x^2-1}}\,dx$

OUR LIST OF SUBSTITUTIONS SAYS WE NEED

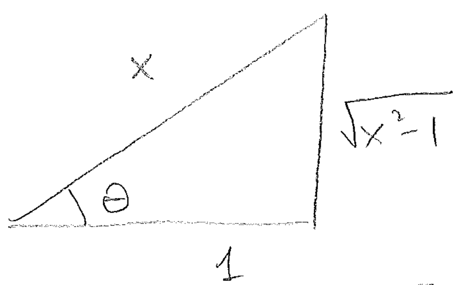
$x = \sec(u)$, SO

$$\int \frac{x^3}{\sqrt{x^2-1}}\,dx = \int \frac{\sec(u)^3}{\sqrt{\tan(u)^2}} \cdot \sec(u)\cdot\tan(u)\,du$$

$$= \int \sec(u)^4\,du \overset{y=\tan(u)}{=} \int (y^2+1)\,y'\,du = \int y^2+1\,dy \Big|_{y=y(u)}$$

$$= \frac{y^3}{3} + y + C \Big|_{y=\tan(u)} = \frac{\tan(u)^3}{3} + \tan(u) + C$$

Now $u = \operatorname{arcsec}(x)$ $\qquad$ ||



$$\frac{\left(\sqrt{x^2-1}\right)^3}{3} + \sqrt{x^2-1} + C$$

EXAMPLE: $\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\,dx$

WE FIRST REWRITE THE 2nd-DEGREE POLYNOMIAL IN ONE OF THE STANDARD FORMS

$$x^2-2x-3=(x-a)^2+b = x^2-2ax+a^2+b$$

So $\quad a=1, \quad b=-3-a^2=-4$

$$\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\,dx = \int_3^5 \frac{\sqrt{(x-1)^2-4}}{x-1}\,dx$$

$$= \int_2^4 \frac{\sqrt{y^2-4}}{y}\,dy = \int_{0=\operatorname{arcsec}(1)}^{\pi/3=\operatorname{arcsec}(2)} \frac{2\sqrt{\sec(u)^2-1}}{\cancel{\sec(u)}}\cdot \cancel{\sec(u)}\operatorname{Tan}(u)\,du$$

$$y=2\sec(u)$$

$$u=\operatorname{arcsec}\left(\frac{y}{2}\right)$$

$$= \int_0^{\pi/3} 2\operatorname{Tan}(u)^2\,du = 2\int_0^{\frac{\pi}{3}} \sec(u)^2-1\,du$$

$$= 2\operatorname{Tan}(u)-u+c\,\Big|_0^{\frac{\pi}{3}} = 2\sqrt{3}-\frac{2}{3}\pi$$

PARTIAL FRACTIONS

WE WANT TO INTEGRATE RATIONAL FUNCTIONS, I.E. QUOTIENTS OF POLYNOMIALS.

WE'RE ALREADY ABLE TO INTEGRATE SOME OF THEM: $\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$

$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\log(x^2+1) + C$, AND ANYTHING IN THE FORM $\frac{P(x)}{x^m}$

EXAMPLE: $\int \frac{2}{9x^2+12x+5}\,dx$

WE WANT TO MAKE $9x^2+12+5$ SOMETHING IN THE FORM $(ax+b)^2+c$

$a=3 \qquad 2abx = 12x$ so $b=2$

$2^2+c=5$ so $c=1$

NOW $\int \frac{2}{9x^2+12x+5}\,dx \underset{u=3x+2}{=} \int \frac{2}{3}\cdot\frac{1}{u^2+1}\,du = \frac{2}{3}\arctan(u)+C$

$= \frac{2}{3}\arctan(3x+2) + C$

EXAMPLE: $\int \frac{8x}{4x^2-4x+5}\,dx = \int \frac{8x}{(2x-1)^2+4}\,dx$