WARM-UP:

EVALUATE $\displaystyle\int_{\pi/4}^{\frac{3}{4}\pi} \cos(x)^4 \, dx$

SOL:

$$\int_{\pi/4}^{\frac{3}{4}\pi} \cos(x)^4 \, dx = \int_{\pi/4}^{\frac{3}{4}\pi} \frac{(1+\cos(2x))^2}{4} \, dx =$$

$$\frac{1}{4} \int_{\pi/4}^{\frac{3}{4}\pi} \cos(2x)^2 + 2\cos(2x) + 1 \, dx =$$

$$\frac{1}{4} \int_{\pi/4}^{\frac{3}{4}\pi} \frac{1+\cos(4x)}{2} \, dx + \frac{1}{2}\int_{\pi/4}^{\frac{3}{4}\pi} \cos(2x) + \frac{\pi}{8} =$$

$$\frac{x}{8} + \frac{\sin(4x)}{32} \Bigg|_{\pi/4}^{\frac{3}{4}\pi} + \frac{\sin(2x)}{4} \Bigg|_{\pi/4}^{\frac{3}{4}\pi} + \frac{\pi}{2} =$$

$$\frac{\pi}{16} - \frac{\sin(\frac{\pi}{2})}{4} + \frac{\sin(\frac{3}{2}\pi)}{4} + \frac{\pi}{8} =$$

$$\frac{3}{16}\pi - \frac{1}{2}$$

TRIGONOMETIC SUBSTITUTION

- SUBSTITUTION AND dx, dy

WE OFTEN LIKE TO "SUBSTITUTE" THE VARIABLE IN OUR COMPUTATIONS, WRITING THINGS SUCH AS $x = \sin(y)$, TO SIMPLIFY THEM. THIS WORKS FINE FOR LIMITS, FOR DERIVATIVES WE HAVE TO USE THE CHAIN RULE APPROPRIATELY, WHAT ABOUT INTEGRALS?

WELL

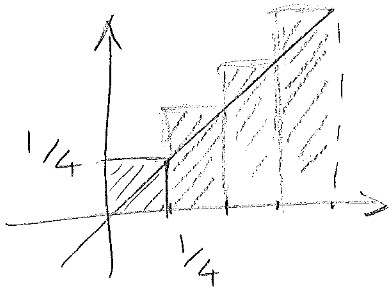
$$\int_0^1 x \, dx = \frac{1}{2}$$ IS DEFINITELY DIFFERENT $(x = y^2)$

FROM $$\int_{0=\sqrt{0}}^{1=\sqrt{1}} y^2 \, dy = \frac{1}{3},$$ SO WHAT

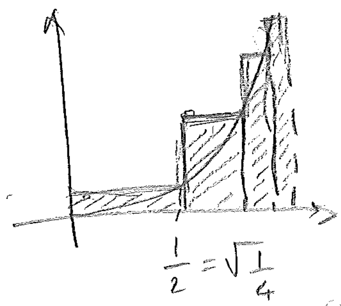
CHANGED? THE TWO FUNCTIONS TAKE ON THE SAME VALUES ON THE RESPECTIVE INTERVAL.

THIS TELLS US THAT THE HEIGHT OF OUR APPROXIMATING RECTANGLES "DOES NOT CHANGE", SO WHAT CHANGED MUST HAVE BEEN THE <u>WIDTH</u>.

$$\int_0^1 x \, dx$$



$$\int_0^1 y^2 \, dy$$



"SAME" RECTANGLES ATER TAKING $y = \sqrt{x}$

SO WHAT CHANGED IS THE WIDTH OF THE APPROXIMATING RECTANGLES.
HOW DOES IT CHANGE?

$$\frac{y(a) \quad y(b)}{a \quad b} = \frac{\text{"y-WIDTH" OF A SMALL RECTANGLE}}{\text{"X-WIDTH" OF A SMALL RECTANGLE}} =$$

$$= \frac{y(b) - y(a)}{b - a} \underset{\substack{\uparrow \\ \text{AS } a \to b}}{=} \frac{dy}{dx}(b) = y'(b)\,!$$

SO IF WE THINK OF dx AS "INFINITESIMAL WIDTH OF RECTANGLES W.R.T. x" AN dy AS "INFINITESIMAL WIDTH OF RECTANGLES W.R.T. $y = y(x)$" WE SEE THAT TO GET THE SAME AREA WE HAVE TO USE

$$dy = dx \cdot y'$$

LET'S TRY: $dy = y' dx \sim dx = \frac{dy}{y'}$

SO $\int_0^1 x\,dx = \int_0^1 \frac{y^2}{y'}\,dy = \int_0^1 \frac{y^2}{(\sqrt{x})'}\,dy =$

$\int_0^1 y^2 \cdot 2\sqrt{x}\,dy \underset{\substack{\uparrow \\ y=\sqrt{x}!}}{=} \int_0^1 2y^3\,dy = \frac{1}{2}$ ✓

WE CAN ALSO THINK OF x AS A FUNCTION OF y! (ON $[0,\infty)$)

THEN $\frac{dx}{dy} = x'(y)$, $dx = x'(y)\,dy$

SO $\int_0^1 x\,dx = \int_0^1 y^2\, x'(y)\,dy \underset{\substack{\uparrow \\ x=y^2 \\ x'=2y}}{=} \int_0^1 2y^3\,dy$

$= \frac{1}{2}$ ✓

TWO WAYS TO USE SUBSTITUTION

WHAT WE DID UP UNTIL NOW IS USE THE SUBSTITUTION RULE THIS WAY:

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du \Big|_{u=u(x)}$$

INPUT OUTPUT

THIS AMOUNTS TO THINKING u AS A FUNCTION OF x.

BUT WE CAN ALSO DO THE OPPOSITE

$x = g(u)$ ($g(u)$ INJECTIVE AND DIFF.)

$$\int f(x)\, dx = \int f(g(u))\, g'(u)\, du \Big|_{u=g^{-1}(x)}$$

INPUT OUTPUT

THIS AMOUNTS TO THINKING OF x AS A FUNCTION OF u

MNEMONIC RULE: u FUNCT. OF x

$$\frac{du}{dx} = u'(x) \longrightarrow du = u'(x)\,dx \sim dx = \frac{du}{u'(x)}$$

X FUNCTION OF $(x = g(u))$

$$\frac{dx}{du} = g'(u) \rightarrow dx = g'(u)\,du \sim du = \frac{dx}{g'(u)}$$

BUT WHY WOULD WE WANT TO USE THE SECOND VERSION? IT LOOKS LIKE IT COMPLICATES THINGS!

EXAMPLE:

$$\int_0^1 \sqrt{1-x^2}\,dx \overset{x=\sin(u)}{=} \int_0^{\pi/2 = \arcsin(1)} \sqrt{1-\sin(u)^2}\cdot\cos(u)\,du =$$

$$= \int_0^{\frac{\pi}{2}} \cos(u)^2\,du = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2u)}{2}\,du =$$

$$\left.\frac{u}{2} + \frac{\sin(2u)}{4}\right|_0^{\frac{\pi}{2}} = \frac{\pi}{4} - 0 + \frac{\sin(0)}{4} - \frac{\sin(\pi)}{4}$$

$$= \frac{\pi}{4} \checkmark$$

SO WE CAN USE IT TO GET RID OF SQUARE ROOTS!

EXAMPLE:

$$\int \sec(x)^4\,dx = \int \sec(x)^2 \sec(x)^2\,dx =$$

$\boxed{u = \operatorname{Tan}(x)}$

$$= \int (u^2+1)\,u'\,dx =$$

$$\int (u^2+1)\,du \Big|_{u=\operatorname{Tan} x} = \frac{u^3}{3} + u \Big|_{u=\operatorname{Tan} x} + C$$

$$= \frac{\operatorname{Tan}(x)^3}{3} + \operatorname{Tan}(x) + C$$

EXAMPLE:

$$\int \operatorname{Tan}(x)^4\,dx = \int \operatorname{Tan}(x)^2 \left(\sec(x)^2 - 1\right) dx$$

$$= \int \operatorname{Tan}(x)^2 \sec(x)^2 - \operatorname{Tan}(x)^2\,dx =$$

$$= \int \operatorname{Tan}(x)^2 \sec(x)^2 - \sec(x)^2 + 1\,dx =$$

$$\int \left(\operatorname{Tan}(x)^2 - 1\right)\sec(x)^2\,dx + \int 1\,dx = \boxed{u = \operatorname{Tan}(x)}$$

$$= \int (u^2-1)\,u'\,dx + x + C = \int u^2 - 1\,du \Big|_{u=\operatorname{Tan}(x)}$$

$$+ x + C = \frac{\operatorname{Tan}(x)^3}{3} - \operatorname{Tan}(x) + x + C$$