PARTIAL FRACTIONS

WE WANT TO INTEGRATE RATIONAL FUNCTIONS, I.E. QUOTIENTS OF POLYNOMIALS.

WE'RE ALREADY ABLE TO INTEGRATE SOME OF THEM: $\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$

$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\log(x^2+1) + C$, AND ANYTHING IN THE FORM $\frac{P(x)}{x^n}$

EXAMPLE: $\int \frac{2}{9x^2+12x+5}\,dx$

WE WANT TO MAKE $9x^2+12+5$ SOMETHING IN THE FORM $(ax+b)^2+c$

$a=3 \qquad 2abx = 12x \quad \text{so } b=2$

$2^2+c=5 \quad \text{so } c=1$

NOW

$$\int \frac{2}{9x^2+12x+5}\,dx \underset{\substack{\uparrow \\ u=3x+2}}{=} \int \frac{2}{3}\cdot\frac{1}{u^2+1}\,du = \frac{2}{3}\arctan(u)+C$$

$$= \frac{2}{3}\arctan(3x+2) + C$$

EXAMPLE: $\int \frac{8x}{4x^2-4x+5}\,dx = \int \frac{8x}{(2x-1)^2+4}\,dx$

WE WANT THE DENOMINATOR TO BE IN THE FORM u^2+1

$$(2x-1)^2+4 = 4\left(\left(\frac{2x-1}{2}\right)^2+1\right) \quad \text{SO}$$

$$u = x-\frac{1}{2}$$

$$\int \frac{8x}{4(u^2+1)}\,dx = \int \frac{2x}{u^2+1}\,dx = \int \frac{2u+1}{u^2+1}\,dx$$

$$\overset{u'=1}{=} \int \frac{2u+1}{u^2+1}\,du \Big|_{u=x-\frac{1}{2}} = \int \frac{2u}{u^2+1}\,du + \int \frac{1}{u^2+1}\,du$$

$$= \log(u^2+1) + \arctan(u) + C = \log\left(\left(x-\tfrac{1}{2}\right)^2+1\right) + \arctan\left(x-\tfrac{1}{2}\right)$$

$+ C$

NOW, MILLION DOLLAR QUESTION:

WOULD YOU RATHER INTEGRATE:

$$\frac{6x^2-22x+18}{x^3-6x^2+11x-6} \quad \text{OR} \quad \frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}$$

ANSWER: YOU ARE LOOKING AT THE SAME FUNCTION!

THAT'S OUR GENERAL IDEA; TO REDUCE TO

$$\frac{1}{x-a}, \quad \frac{1}{x+a^2}, \quad \frac{x}{x^2+a}$$ AND THEIR COMBINATIONS. (PLUS A POLYNOMIAL)

X^2+a^2 IS A POLYNOMIAL WITH NO ROOTS IN $\mathbb{R}$, THUS IRREDUCIBLE. WHAT IF THE DENOMINATOR IS REDUCIBLE?

EXAMPLE: FIND A, B SUCH THAT

$$\frac{4x+5}{(x+1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(2x+3)}$$

$$\sim \quad 4x+5 = A(2x+3) + B(x+1)$$

$$\sim \quad 4x+5 = 2Ax + 3A + Bx + B$$

$$\sim \quad (4-2A-B)x + 5 - 3A - B = 0$$

SO $\begin{cases} 4-2A-B=0 \\ 5-3A-B=0 \end{cases} \sim \begin{cases} 4-2A-B=0 \\ 1-A=0 \end{cases}$

$$\sim \begin{cases} B=2 \\ A=1 \end{cases}$$

CHECK

$$\frac{1}{x+1} + \frac{2}{2x+3} = \frac{2x+3+2(x+1)}{(x+1)(2x+3)} = \frac{4x+5}{(x+1)(2x+3)} \quad \checkmark$$

NOW $\displaystyle\int \frac{4x+5}{(x+1)(2x+3)}\,dx = \log(x+1) + \log(2x+3) + C$

OPTION 2 (SMARTER, EASIER): <u>RESIDUES</u>

SAY WE HAVE

$$\frac{4x+5}{(x+1)(2x+3)} = \frac{A}{x+1} + \frac{B}{2x+3}$$

THEN

$$\frac{4x+5}{2x+3} = A + \frac{(x+1)B}{2x+3}$$

EVALUATE AT $x=-1$

$$1 = \frac{-4+5}{1} = A$$

TO FIND B

$$\frac{4x+5}{x+1} = \frac{A(2x+3)}{x+1} + B$$

EVALUATE AT $x = -\frac{3}{2}$

$$2 = \frac{-6+5}{-\frac{1}{2}} = B \qquad !!$$

QUESTION: HOW CAN WE REWRITE

$$\frac{12x+30}{4x^3-4x^2-15x} = \frac{12x+30}{x(2x+3)(2x-5)} \quad ?$$