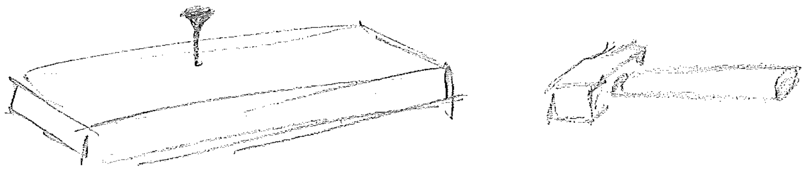
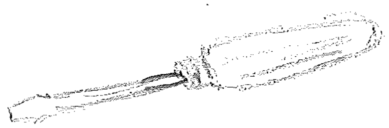
INTEGRATION BY PARTS

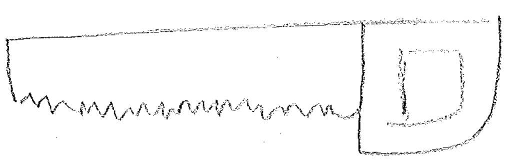
IN OUR QUEST TO UNDERSTANDING INTEGRATION, WE HAVE DEVELOPED THE "BASIC MATERIALS": FTC



A "SCREWDRIVER": SUBSTITUTION



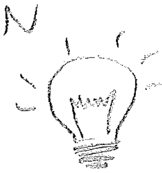
WE WILL NOW DEVELOP A "HACKSAW":



INTEGRATION BY PARTS.

WE WANT TO SOLVE INTEGRALS SUCH AS $\int x e^x \, dx$.

OUR GENERAL IDEA UP TO NOW HAS BEEN



DERIVATION RULE $\longrightarrow$ INTEGRATION RULE

ARE THERE ANY (BASIC) DERIVATION RULES LEFT?

THE PRODUCT RULE:

$$\frac{d}{dx} u(x)v(x) = \left(\frac{d}{dx} u(x)\right)v(x) + u(x)\left(\frac{d}{dx} v(x)\right)$$

WELL, LET'S TRY INTEGRATING IT

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

$$u(x)v(x) - \int u'(x)v(x)\,dx = \int u(x)v'(x)\,dx$$

CAN WE APPLY IT TO $\int xe^x$

WELL, WE'D LIKE TO DERIVE x...

$u(x) = x$, $v'(x) = e^x$ (SO $v(x) = e^x$)

THEN

$$xe^x - \int x'e^x\,dx = \int xe^x\,dx \sim$$

$$xe^x - \int e^x\,dx = \int xe^x\,dx \sim xe^x - e^x + C =$$

$\int xe^x dx$! SO $\int xe^x dx = (x-1)e^x + C$!

INTEGRATION BY PARTS:

LET $F(x), G(x)$ BE DIFFERENTIABLE FUNCTIONS, WITH $\frac{d}{dx}F(x) = f(x)$

AND $\frac{d}{dx}G(x) = g(x)$

THEN

$$\underline{F(x)G(x) - \int f(x)G(x)\,dx = \int F(x)g(x)}$$

DEFINITE VERSION:

$$\underline{F(x)G(x)\Big|_a^b - \int_a^b f(x)G(x)\,dx = \int_a^b F(x)g(x)\,dx}$$

SO HOW DO WE USE IT? WHEN WE SEE AN INTEGRAL OF A PRODUCT OF FUNCTIONS, WE ASK OURSELVES: WOULD I BE BETTER OFF WITH THE DERIVATIVE OF ONE AND THE INTEGRAL OF THE OTHER?

<u>EXAMPLE</u>: $\int x \sin x\,dx$

OUR FAVOURITE FUNCTION TO DERIVE: x!

$F(x) = x \qquad g(x) = \sin(x)$

THEN

$$\int x \sin(x)\,dx = -\int 1\cdot(-\cos(x))\,dx + (x\cdot(-\cos(x)))$$

$$\int F(x)g(x) = -\int f(x)G(x)\,dx + F(x)G(x)$$

SO $\int x \sin x\,dx = \int \cos(x)\,dx - x\cos x =$

$= \sin x - x\cos x + C$

EXAMPLE (HIGHER POWERS OF x): $\int x^2 e^x\,dx$

AGAIN, WE'D LIKE TO DERIVE POWERS OF x SO THE DEGREE GOES DOWN

$F(x) = x^2,\ g(x) = e^x$

JUST COMPUTED!

$$\int x^2 e^x\,dx = x^2 e^x - \int 2x e^x\,dx = x^2 e^x - 2(x-1)e^x + C$$

$= (x^2 - 2x - 2)e^x + C$

LET'S CHECK OUR ANSWER:

$$\frac{d}{dx}(x^2-2x-2)e^x = (2x-2)e^x + (x^2-2x-2)e^x = x^2 e^x \checkmark$$

EXAMPLE: $\int x \log x \, dx$

TWO OPTIONS:

$F(x) = x$, $g(x) = \log x$ AND WE HAVE TO COMPUTE

$$\int \left(\int \log x \, dx \right) dx$$

OR

$F(x) = \log x$, $g(x) = x$ AND WE HAVE TO COMPUTE

$$\int \frac{x^2}{2} \cdot \frac{1}{x} \, dx \quad \leftarrow \text{MUCH BETTER!}$$

SO FOR ONCE WE WANT TO INTEGRATE, NOT DERIVE x

$$\int x \log x \, dx = \frac{x^2}{2} \log x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx =$$

$$\frac{x^2}{2} \log x - \frac{x^2}{4} + C.$$

EXAMPLE: $\int \log x \, dx$

LAST TIME WE "GUESSED" IT, SO LET'S USE I.B.P. TO ACTUALLY COMPUTE IT.

- BUT IT'S NOT A PRODUCT!
- SURE IT IS. $\log x = 1 \cdot \log x$,

$F(x) = \log x$, $g(x) = 1$

$$\int \log x\, dx = x \log x - \int x \cdot \frac{1}{x}\, dx = x \log x - x + c \,!$$

EXAMPLE: $\int \arctan(x)\, dx$

SAME TRICK: $\arctan(x) = \arctan(x) \cdot 1$

$F(x) = \arctan(x)$, $g(x) = 1$

$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx$$

$$\underset{\substack{\uparrow \\ \text{SUBSTITUTION} \\ 1+x^2=u}}{=} x \arctan x - \int \frac{u'}{u}\, dx = x \arctan x - \int \frac{1}{u}\, du \Big|_{u=1+x^2}$$

$$= x \arctan x - \log(1+x^2)$$

CHECK: $\frac{d}{dx}\left(x \arctan x - \log(1+x^2)\right) =$

$$\arctan x + \frac{x}{1+x^2} - \frac{x}{1+x^2} = \arctan x \checkmark$$