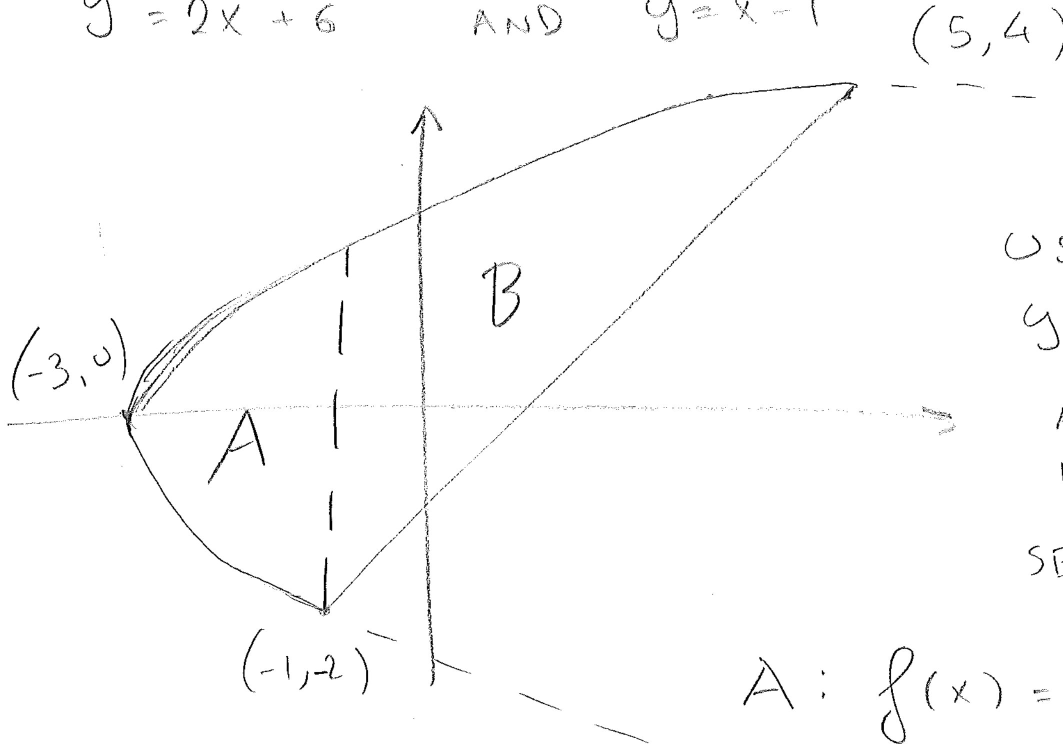
WARM-UP

FIND THE AREA BOUNDED BY

$y^2 = 2x + 6$ AND $y = x - 1$



USE $y = \pm\sqrt{2x+6}$ AN SPLIT IN TWO SECTORS

A: $f(x) = \sqrt{2x+6}$

$g(x) = -\sqrt{2x+6}$

B: $f(x) = \sqrt{2x+6}$ $\quad g(x) = x - 1$

SOL:

$u = 2x + 6$

A: $\displaystyle\int_{-3}^{-1} 2\sqrt{2x+6}\,dx = \int_0^4 \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_0^4 = \frac{16}{3}$

B: $\displaystyle\int_{-1}^{5} \sqrt{2x+6} - x + 1\,dx = \frac{1}{2}\int_4^{16}\sqrt{u}\,du - \int_{-1}^{5} x - 1\,dx$

$\displaystyle = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_4^{16} - \left(\frac{x^2}{2} - x\right)\Big|_{-1}^{5} = \frac{64}{3} - \frac{8}{3} - \frac{15}{2} + \frac{3}{2} = -6 + \frac{56}{3}$

$A + B = \frac{56}{3} + \frac{16}{3} - 6 = \frac{72}{3} - 6 = 24 - 6 = 18$

(SEE APPROACH 2 NEXT PAGE)

ONE MORE INTRICATE EXAMPLE:

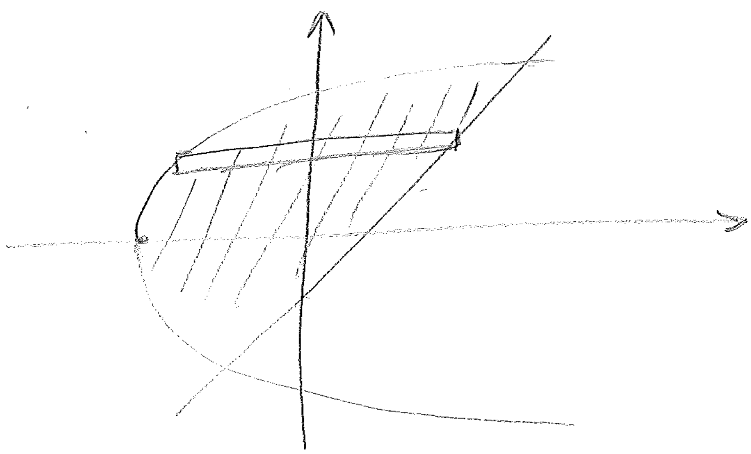
CURVES DO NOT NEED TO BE IN THE FORM $y = f(x)$!

EXAMPLE:

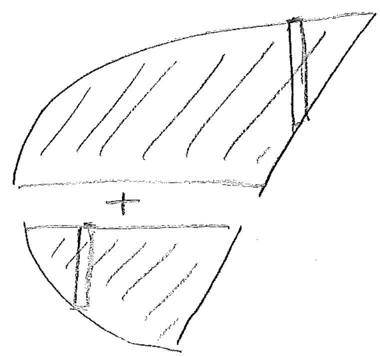
FIND THE AREA ENCLOSED BY

$y^2 = 2x+6$ AND $y = x-1$

SKETCH: $y^2 = 2x+6 \sim x = -3 + \frac{y^2}{2}$, A PARABOLA



APPROACH 1:
SPLIT AS



APPROACH 2: y AS A VARIABLE!

$x = \underbrace{\frac{y^2}{2} - 3}_{T(y)}$ $\quad x = \underbrace{y+1}_{B(y)}$

$T(y) = B(y) \sim \frac{y^2}{2} - 3 = y+1 \sim y^2 - 2y - 8 = 0$

$y = -2, 4$

$$\int_{-2}^{4} B(y) - T(y)\,dy = \int_{-2}^{4} -\frac{y^2}{2} + y + 4\,dy = -\frac{y^3}{6} + \frac{y^2}{2} + 4y \Big|_{-2}^{4}$$

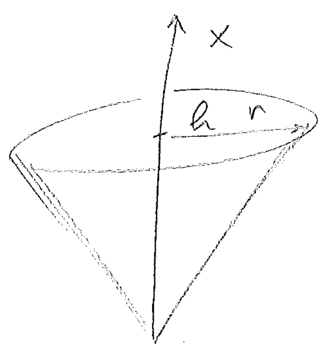
$$= \left(-\frac{64}{6} + \frac{16}{2} + 16\right) - \left(+\frac{8}{6} + \frac{4}{2} - 8\right) = 18$$

VOLUMES:

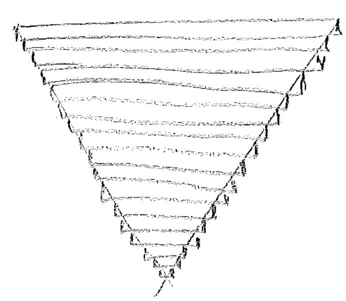
WE MOVE ON TO THE REALM OF THREE DIMENSIONAL SHAPES.

MOST THREE DIMENSIONA SHAPES REQUIRE MULTI-VARIABLE INTEGRALS TO COMPUTE, BUT IN SOME CASES WE CAN USE THE SIMMETRIES OF THE PROBLEM TO REDUCE TO ONE VARIABLE.

EXAMPLE: CONE OF HEIGHT h, RADIUS r



WE SLICE (APPROXIMATE) THE CONE INTO "PANCAKES"

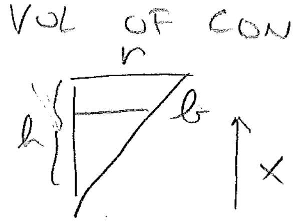


AS THE NUMBER OF PANCAKES GOES TO ∞

VOL OF PANCAKE $= \Delta_x$ AREA

$\sum$ VOL OF PANC. $= \int$ AREA dx = VOL OF CONE

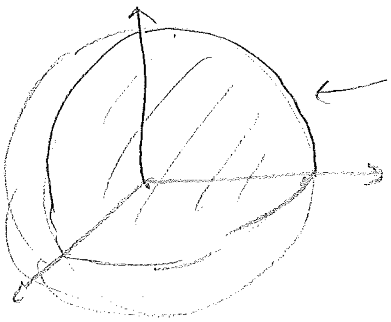
AREA OF SLICE AT HEIGHT x =



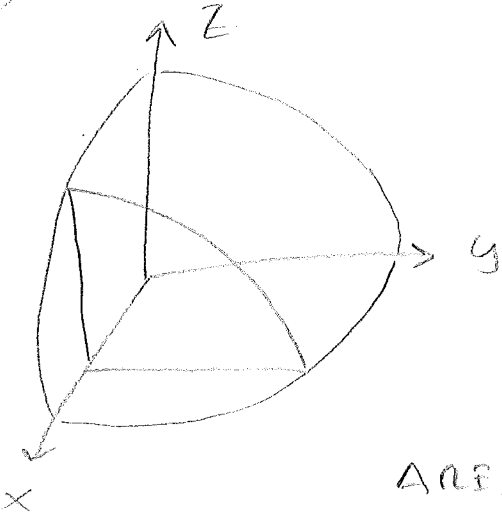
$$\frac{r}{h} = \frac{b}{x} \sim b = \frac{xr}{h} \qquad \text{AREA} = \pi \frac{x^2 r^2}{h^2}$$

$$\text{VOL} = \int_0^h \pi \frac{x^2 r^2}{h^2}\, dx = \frac{\pi r^2}{h^2}\left(\frac{x^3}{3}\right)\Big|_0^h = h r^2 \cdot \frac{\pi}{3}$$

EXAMPLE: SPHERE OF RADIUS r



← WE'LL COMPUTE THIS AND MULTIPLY BY 8



FIX A VALUE OF X

$$x^2+y^2+z^2=r^2$$

$$\underbrace{y^2+z^2}_{\text{CIRCLE!}}=\underbrace{r^2-x^2}_{\text{CONSTANT}}$$

RADIUS $=\sqrt{r^2-x^2}$

AREA $=(r^2-x^2)\cdot\pi$ $\left(\cdot\frac{1}{4}\right.$ AS WE ARE ONLY TAKING  $\left.\right)$

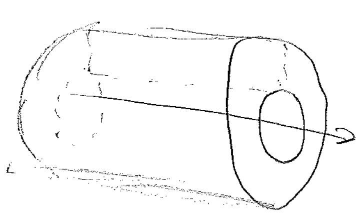
SO VOL OF OCTANT =

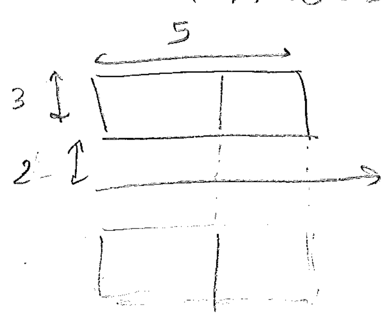
$$\int_0^r \frac{(r^2-x^2)\pi}{4}\,dx=\frac{\pi}{4}\left(xr^2-\frac{x^3}{3}\right)\Big|_0^r=\frac{\pi}{4}\cdot\frac{2}{3}r^3=\frac{\pi}{6}r^3$$

VOL OF SPHERE $8\cdot\frac{\pi}{6}r^3=\frac{4}{3}\pi r^3$

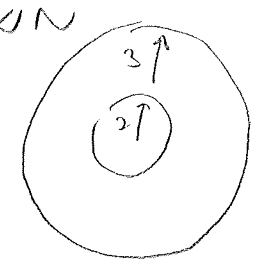
EXAMPLE: REVOLVING A REGION

WE CONSTRUCT AN HOLLOW CYLINDER BY REVOLVING A RECTANGLE AROUND THE X-AXIS





SLICE: A CIRCULAR CROWN

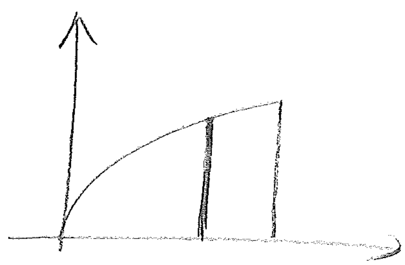


AREA: $5^2\cdot\pi-2^2\cdot\pi=21\pi$

SO VOLUME $\displaystyle\int_0^5 21\pi \, dx = 21\pi x \Big|_0^5 = 105$

EXAMPLE: MORE REVOLVING

WE REVOLVE THE REGION BETWEEN $y = \sqrt{x}$, $y = 0$ AND $x = 4$ AROUND $y = 0$



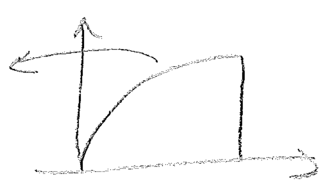
SLICE = CIRCLE OF RADIUS $\sqrt{x}$

AREA OF SLICE $= \pi(\sqrt{x})^2 = \pi x$

VOLUME $\displaystyle\int_0^4 \pi x \, dx = \frac{\pi x^2}{2}\Big|_0^4 = 8\pi$

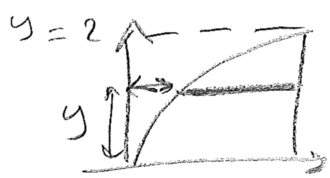
EXAMPLE: EVEN MORE REVOLVING

SAME REGION IS ROTATED AROUND $x = 0$



WE PICK <u>HORIZONTAL</u> SLICES

(WE ALWAYS WANT FLAT PANCAKES)



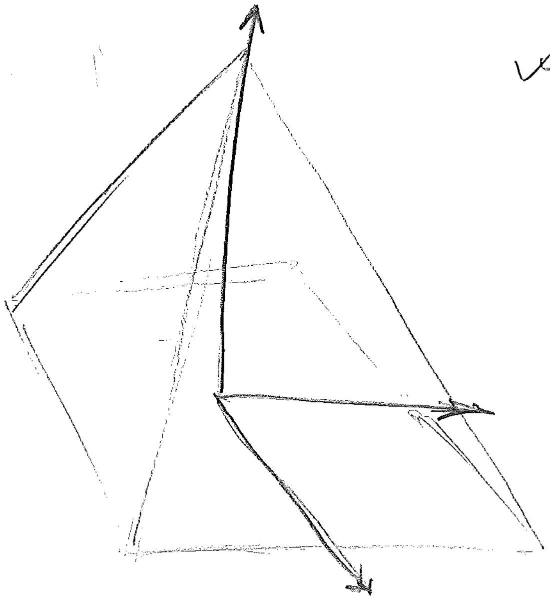
SLICE = CIRCULAR CROWN,

$R = 4$, $r = y^2$ $\left(\sqrt{x} = y \Rightarrow y^2 = x\right)$

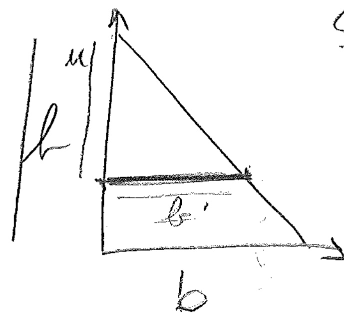
AREA OF SLICE IS $\pi R^2 - \pi r^2 = (16 - y^4)\pi$

VOL: $\displaystyle\pi\int_0^2 16 - y^4 \, dy = \pi\left(16y - \frac{y^5}{5}\right)\Big|_0^2 = \frac{128\pi}{5}$

EXAMPLE: SQUARE PYRAMID BASE 2b, HEIGHT h



WE PICK ONE OCTANT



SLICE: SQUARE

$$\frac{b}{h} = \frac{b'}{u} \qquad b' = \frac{u \cdot b}{h}$$

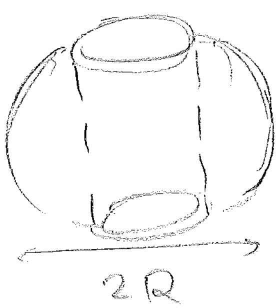
$$(u = z - h)$$

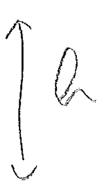
AREA OF SLICE $(b')^2 = u^2 \cdot \frac{b^2}{h^2}$

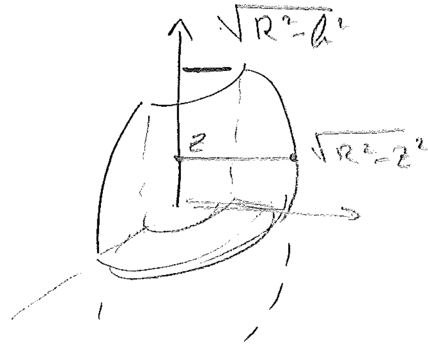
$$\frac{1}{4}\,\text{VOL} = \int_0^h u^2 \cdot \frac{b^2}{h^2}\, du = \frac{u^3}{3} \cdot \frac{b^2}{h^2}\Bigg|_0^h = \frac{h \cdot b^2}{3}$$

$$\boxed{\text{VOL} = \frac{4}{3} h b^2}$$

EXAMPLE: NAPKIN RING







SLICE: CIRCULAR CROWN

$$R = \sqrt{R^2 - z^2}$$

$$r = \sqrt{R^2 - h}$$

AREA OF SLICE $= \pi(R^2 - R^2 - z^2 + h^2) = \pi(h^2 - z^2)$

$$\text{VOL}: \; 2\int_0^h \pi(h^2 - z^2)\, dz = 2\pi\left(h^2 z - \frac{z^3}{3}\right)\Bigg|_0^h = \pi \cdot \frac{4}{3} h^3$$