WARM-UP:

① USE SUBSTITUTION TO COMPUTE

$$\int \tan(x)\,dx$$

② USE INTEG. BY PARTS TO COMPUTE

$$\int \arcsin(3x)\,dx$$

SOL:

① $\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}\,dx \qquad u = \cos(x)$

$= -\int \frac{u'}{u}\,dx = -\int \frac{1}{u}\,du \Big|_{u=\cos(x)} =$

$-\log(\cos(x)) + C = \log(\sec(x)) + C$

② $\int \arcsin(3x)\,dx = \int 1 \cdot \arcsin(3x)\,dx$

$= x\arcsin(3x) - \int \frac{3x}{\sqrt{1-9x^2}}\,dx =$

$u = 1 - 9x^2$

$x\arcsin(3x) + \frac{1}{3}\int \frac{u'}{\sqrt{u}}\,dx =$

$x\arcsin 3x + \frac{2}{3}\sqrt{1-9x^2}$

TRIGONOMETRIC INTEGRALS

TRIG FUNCTIONS ARE POSSIBLY THE MOST UBIQUITOUS FUNCTIONS IN PHYSICS AND ENGINEERING. IT'S ONLY NATURAL THAT WE'D WANT TO HAVE AN ALGORITHM TO INTEGRATE SIMPLE COMBINATIONS OF THESE FUNCTIONS.

FIRST WE HAVE TO RECALL SOME IMPORTANT TRIGONOMETRIC IDENTITIES:

- $\sin(x)^2 + \cos(x)^2 = 1$
- $\sin(2x) = 2\sin x \cos x$
- $\cos(2x) = \cos(x)^2 - \sin(x)^2 = 2\cos(x)^2 - 1 = 1 - 2\sin(x)^2$
- $\sin(x)^2 = \dfrac{1-\cos(2x)}{2}$
- $\cos(x)^2 = \dfrac{1+\cos(2x)}{2}$

THE LAST TWO ALLOW US TO REWRITE HIGHER POWERS OF $\sin(x)$, $\cos(x)$ IN TERMS OF LOWER POWERS

EXAMPLE: $\cos(x)^4 = \left(\dfrac{1+\cos(2x)}{2}\right)^2 = \dfrac{1}{4}\left(\underbrace{\cos(2x)^2} + 2\cos(2x) + 1\right)$

$= \dfrac{1}{4}\left(\underbrace{\dfrac{1+\cos(4x)}{2}} + 2\cos 2x + 1\right) = \dfrac{\cos(4x)}{8} + \dfrac{\cos(2x)}{2} + \dfrac{3}{8}$

SO WHILE WE WOULD HAVE TO COME UP WITH SOME IDEA TO COMPUTE $\int \cos(x)^4$ DIRECTLY, IT IS IMMEDIATE USING THE IDENTITY ABOVE.

WE'LL USE SIMILAR TRICKS, IN ADDITION TO THE INTEGRATION TECHNIQUES WE DEVELOPED TO SOLVE TWO FAMILY OF INTEGRALS, THA IS

$$\int \sin(x)^m \cos(x)^n \, dx \quad \text{AND}$$

$$\int \tan(x)^m \sec(x)^n \, dx$$

FOR INTEGERS n, m. THE DETAILS WILL DEPEND ON n, m BEING ODD OR EVEN.

THE INTEGRAL OF $\sin(x)^m \cos(x)^n$

CASE 1: ONE OF THEM IS ODD (EXACTLY ONE)

EXAMPLE: $$\int \underbrace{\sin(x)}_{u}{}^2 \underbrace{\cos(x)}_{u'} \, dx = \int u^2 u' \, dx \Big|_{u=\sin(x)}$$

$$= \int u^2 \, du \Big|_{u=\sin x} = \frac{u^3}{3} + C \Big|_{u=\sin(x)} = \frac{\sin(x)^3}{3} + C$$

THIS IDEA ALWAYS WORKS:

- PICK THE EVEN ONE
- WRITE $u = \sin(x)$ OR $\cos(x)$, WHICHEVER IS THE EVEN ONE (LET'S SAY IT'S $\sin(x)$ FOR SIMPLICITY)
- NOW WE HAVE $\sin(x)^m \cos(x)^n =$

$$\underbrace{u^m}_{\sin^m} \cdot \underbrace{u'}_{\cos} \cdot \underbrace{(1-u^2)^{\frac{n-1}{2}}}_{\cos^{n-1} \leftarrow \text{EVEN!}}$$

- NOW $\int \sin(x)^m \cos(x)^n \, dx = \int u^m (1-u^2)^{\frac{n-1}{2}} \cdot u' \, dx$

$$= \int u^m (1-u^2)^{\frac{n-1}{2}} \, du \Big|_{u = \sin x}$$

EXAMPLE:

$$\int \sin(x)^3 \cos(x)^2 \, dx \qquad \begin{aligned} u &= \cos(x) \\ u' &= -\sin(x) \end{aligned}$$

$$= \int -u' \cdot u^2 \cdot (1-u^2) \, dx = \int -u^2(1-u^2) \, du \Big|_{u=\cos(x)}$$

$$= \int u^4 - u^2 \, du \Big|_{u=\cos x} = \frac{u^5}{5} - \frac{u^3}{3} \Big|_{u=\cos x} + C$$

$$= \frac{\cos(x)^5}{5} - \frac{\cos(x)^3}{3} + C$$

BOTH m AND m ARE EVEN

IDEA: WE WANT TO USE TRIG IDENTITIES TO GET BACK TO PREVIOUS CASE

EXAMPLE:

$$\int \cos(x)^2 dx = \int \frac{1+\cos(2x)}{2} dx = \frac{2X+\sin(x)}{4} + C$$

EXAMPLE:

$$\int \cos(x)^2 \sin(x)^2 dx = \int \frac{1+\cos(2x)}{2} \cdot \frac{1-\cos(2X)}{2} dx$$

$$= \frac{1}{4}\int 1-\cos^2(2x)\, dx = \frac{1}{4}\int 1 - \frac{1+\cos(4x)}{2} dx$$

$$= \int \frac{1}{8} dx - \int \frac{\cos(4x)}{8} dx = \frac{X}{8} - \frac{\sin(4X)}{32} + C$$

SO, A MORE PRECISE ALGORITHM IS:

- USE $\sin(x)^2 = \frac{1-\cos(2x)}{2}$, $\cos(x)^2 = \frac{1+\cos(2x)}{2}$ TO DIVIDE THE DEGREE BY 2, EXPAND NEW EXPRESSION

- FOR ODD PARTS, GO BACK TO PREV ALGORITHM. FOR EVEN PARTS, USE IDENTITIES AGAIN.

THE CASE m, m ODD IS HARDER; WE WON'T CONSIDER IT.

THE INTEGRAL OF $\mathrm{Tan}(x)^m \sec(x)^n$

RECALL THAT

$\sec(x) = \frac{1}{\cos(x)}$, $(\mathrm{Tan}(x))' = \sec(x)^2$,

$\sec(x)' = \sec(x)\,\mathrm{Tan}(x)$ $\quad 1+\mathrm{Tan}(x)^2 = \sec(x)^2$

THERE ARE FIVE BASIC CASES:

1) m ODD, n ANY (EVEN FRACTIONAL!)

$$(\mathrm{Tan}(x))^m (\sec(x))^n = \frac{\sin(x)^m}{\cos(x)^{m+n}} = \frac{\sin(x)^{m-1}}{\cos(x)^{m+n}} \sin(x)$$

$u = \cos(x)$

2) ALTERNATIVELY m ODD, $n \geq 1$

$$\mathrm{Tan}(x)^m \sec(x)^n = \mathrm{Tan}(x)^{m-1} \sec(x)^{n-1} \cdot \sec(x)\,\mathrm{Tan}(x)$$

$u = \sec(x)$ $\quad u' = \mathrm{Tan}(x)\sec(x)$, $\mathrm{Tan}(x)^{m-1} =$
$(\sec(x)^2 - 1)^{\frac{m-1}{2}}$

3) n EVEN ≥ 2 $\quad \mathrm{Tan}(x)^m \sec(x)^n =$

$\mathrm{Tan}(x)^m \sec(x)^{n-2} \sec(x)^2$ $\qquad u = \mathrm{Tan}(x)$
$\qquad u' = \sec(x)^2$

THEN USE $\sec(x)^2 = \mathrm{Tan}(x)^2 + 1$

4) m EVEN, $n = 0$ $\quad \mathrm{Tan}(x)^m = \mathrm{Tan}(x)^{m-2} (\sec(x)^2 - 1)$

REPEATEDLY AND THEN $u = \mathrm{Tan}(x)$

(5) m ODD, m EVEN.

$//$

TOO COMPLICATED FOR NOW. WE'LL GET BACK TO IT LATER IN THE COURSE.

AN EXAMPLE FOR EACH CASE:

EXAMPLE:

$$\int \tan(x)^3 \sec(x)^{\frac{1}{2}}\,dx = \int \frac{\sin(x)^3}{\cos(x)^{\frac{7}{2}}}\,dx$$

$u = \cos(x)$

$$\int \frac{\sin(x)^3}{\cos(x)^{\frac{7}{2}}}\,dx = \int \frac{-(1-u^2)}{u^{\frac{7}{2}}}\cdot u'\,dx \Big|_{u=\cos(x)}$$

$$= -\int u^{-\frac{7}{2}} - u^{-\frac{3}{2}}\,du \Big|_{u=\cos(x)} = \frac{2}{7}u^{-\frac{5}{2}} + 2u^{-\frac{3}{2}} + C \Big|_{u=\cos x}$$

$$= \frac{2}{7}\cos(x)^{-\frac{5}{2}} + 2\cos(x)^{-\frac{3}{2}} + C$$

EXAMPLE:

$$\int \tan(x)^3 \sec(x)^4\,dx = \int \tan(x)^2 \sec(x)^2 \cdot \tan(x)\sec(x)^2\,dx$$

$\boxed{u = \sec(x)}$ $$= \int (u^2-1)^2 u^3 \cdot u'\,dx = \int (u^2-1)^2 u^3\,du \Big|_{u=\sec x}$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C \Big|_{u=\sec(x)} = \frac{\sec(x)^6}{6} - \frac{\sec(x)^4}{4} + C$$

EXAMPLE:

$$\int \sec(x)^4\,dx = \int \sec(x)^2 \sec(x)^2\,dx =$$

$\boxed{u = \tan(x)}$

$$= \int (u^2+1)\,u'\,dx =$$

$$\int (u^2+1)\,du \Big|_{u=\tan x} = \frac{u^3}{3} + u \Big|_{u=\tan x} + C$$

$$= \frac{\tan(x)^3}{3} + \tan(x) + C$$

EXAMPLE:

$$\int \tan(x)^4\,dx = \int \tan(x)^2\left(\sec(x)^2 - 1\right)dx$$

$$= \int \tan(x)^2 \sec(x)^2 - \tan(x)^2\,dx =$$

$$= \int \tan(x)^2 \sec(x)^2 - \sec(x)^2 + 1\,dx =$$

$$\int \left(\tan(x)^2 - 1\right)\sec(x)^2\,dx + \int 1\,dx = \boxed{u = \tan(x)}$$

$$= \int (u^2-1)\,u'\,dx + x + C = \int u^2 - 1\,du \Big|_{u=\tan(x)}$$

$$+ x + C = \frac{\tan(x)^3}{3} - \tan(x) + x + C$$