WARM-UP:

SOLVE THE FOLLOWING INDEFINITE INTEGRALS

① $\int (4x+1)^{\frac{3}{2}} dx$

② $\int (3-\cos(x))\, e^{\sin(x)-3x} dx$

SOL:

① WE KNOW HOW TO FIND $\int x^{\frac{3}{2}} dx$,

SO LET $u(x) = 4x+1$

THEN $u'(x) = 4$, AND

$$\int (4x+1)^{\frac{3}{2}} dx = \int \frac{1}{4} u' \cdot u^{\frac{3}{2}} dx =$$

$$\int \frac{1}{4} u^{\frac{3}{2}} du \Big|_{u=4x+1} + C = \frac{1}{4} \cdot \frac{2}{5} u^{\frac{5}{2}} \Big|_{u=4x+1} + C = \frac{(4x+1)^{\frac{5}{2}}}{10} + C$$

② $(\sin(x)-3x)' = \cos(x) - 3$ SO IF $u = \sin x - 3x$

THEN $(3-\cos x)\, e^{\sin x - 3x} = -u' e^{u}$

SO $\int (3-\cos(x))\, e^{\sin x - 3x} dx = \int -u' e^{u} dx$

$$= -\int e^{u} du \Big|_{u=\sin x - 3x} = -e^{\sin x - 3x} + C$$

WHEN WE APPLY THE SUBTITUTION RULE TO DEFINITE INTEGRALS WE NEED TO BE A <u>BIT MORE CAREFUL</u>

SUBS. RULE, DEFINITE VERSION

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

SO AFTER WE APPLY SUBSTITUTION THE VARIABLE u TAKES ON ALL VALUES FROM $u(a)$ TO $u(b)$.

TWO METHODS TO SOLVE SAME INTEGRAL:

- USE SUBSTITUTION TO FIND

  $\int f(u(x))\,u'(x)\,dx$ (FUNCTION OF x)

  AND EVALUATE AT $x=a$, $x=b$

- FIND

  $\int f(u)\,du$ (FUNCTION OF u) AND

  EVALUATE AT $u=u(a)$ $u=u(b)$

THE SECOND OPTION IS OFTEN EASIER FOR DEFINITE INTEGRALS

EXAMPLE:

- $\int_0^1 x^2 \sin(x^3+1)\,dx$ WE TRY PICKING "$u'(x) = x^2$"

  FACTORS

NOW, $\sin(x^3+1) = \sin(u)$ WITH $u = x^3+1$;

$u' = 3x^2$ WHICH IS "ALMOST" x^2; SO

WITH SOME TINKERING

$$\int_0^1 x^2 \sin(x^3+1)\,dx = \frac{1}{3}\int_0^1 3x^2 \sin(x^3+1)\,dx$$

$$= \frac{1}{3}\int_0^1 u'(x)\sin(u(x))\,dx = \frac{1}{3}\int_{u(0)}^{u(1)} \sin(u)\,du$$

$$= \frac{1}{3}\int_1^2 \sin(u)\,du = \frac{1}{3}\left(-\cos(u)\right)\Big|_1^2 = \frac{1}{3}\cdot\left(\cos(1)-\cos(2)\right)$$

RECALL
$u = x^3+1$

- $\int_0^1 \frac{x}{1+x^2}\,dx$ WE REWRITE THE INTEGRAND AS $x \cdot \frac{1}{1+x^2}$

IF $u = 1+x^2$ THEN $u' = 2x$ AND

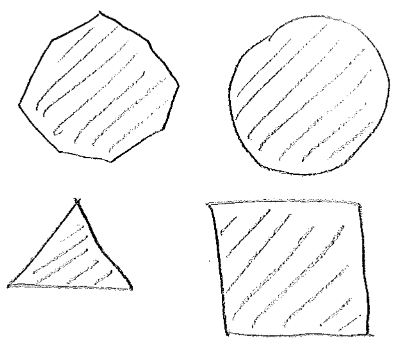
$x \cdot \frac{1}{1+x^2} = \frac{1}{2}\cdot u' \cdot \frac{1}{u}$ SO

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{u=1}^{u=2} \frac{1}{u}\,du = \frac{\log(u)}{2}\Big|_1^2 = \frac{\log 2}{2}$$

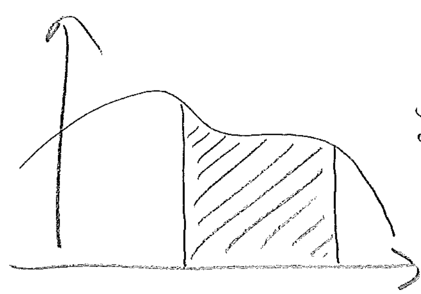
AREA BETWEEN CURVES

BEFORE WE DEVELOP SOME NEW INTEGRATION TOOL, WE ARE NOW ABLE TO EXPLORE SOME APPLICATIONS OF DEFINITE INTEGRALS.

ONE IMPORTANT QUESTION IN CALCULUS IS HOW DO WE COMPUTE AREAS? FOR NOW WE KNOW



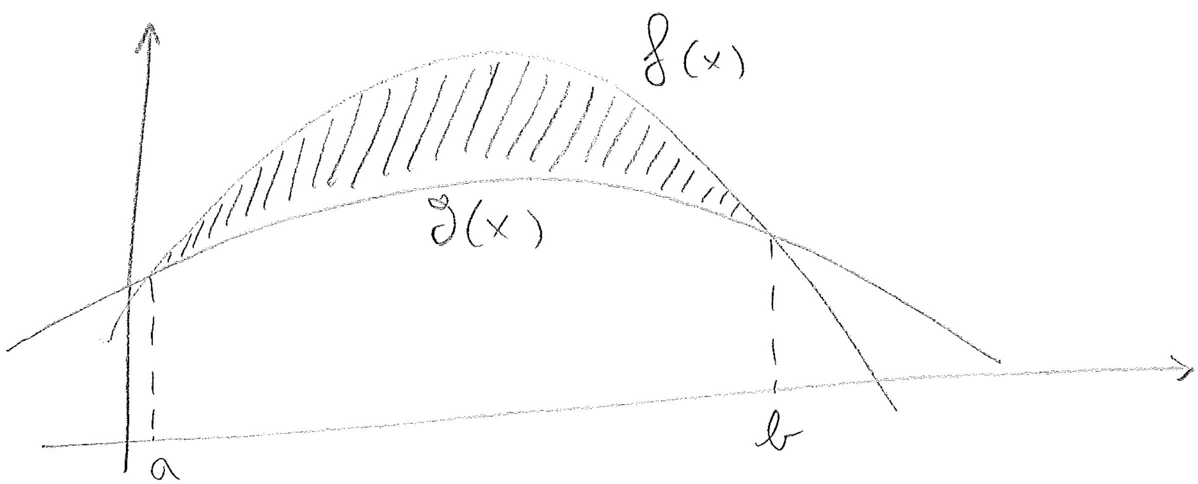
CLASSICAL PLANE AREAS



$$\int_a^b f(x)\,dx$$

AREA UNDER A CURVE

A SMALL, BUT POWERFUL GENERALIZATION IS:



THE AREA BETWEEN CURVES.

IN THE CASE ABOVE, AS WE KNOW THAT $f(x) \geq g(x)$ ON $[a, b]$, AND BOTH ARE POSITIVE, WE SEE THAT THE AREA MUST BE

$$\text{AREA BELOW } f(x) - \text{AREA BELOW } g(x) =$$
$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

EXAMPLE:

$$f(x) = 16 - 5(x-2)^2, \quad g(x) = 4 - 2(x-2)^2$$

$f(x)$ AND $g(x)$ INTERSECT AT THE SOLNS OF

$$f(x) - g(x) = 0 \sim 12 - 3(x-2)^2 = 0$$

$$\sim x = 0, 4$$

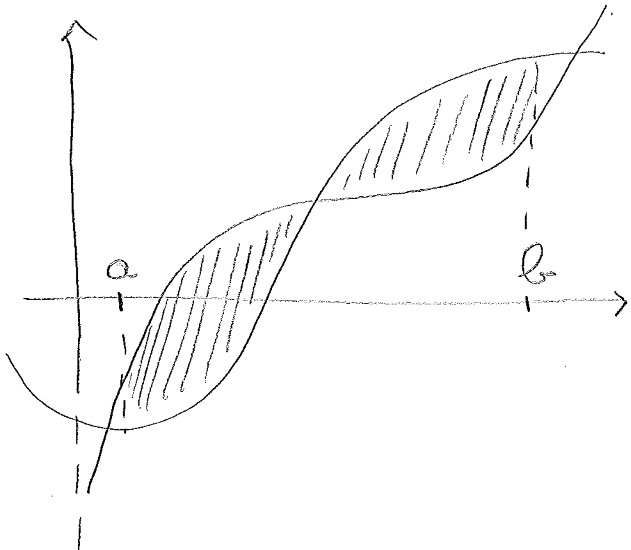
AT $x = 2$ $f(2) > g(2)$ SO THE AREA BETWEEN THEM (AS IN FIGURE) IS

$$\int_0^4 f(x) - g(x)\,dx = \int_0^4 12 - 3(x-2)^2\,dx \overset{u'=1}{=} \int_{-2}^{2} 12 - 3u^2\,du$$

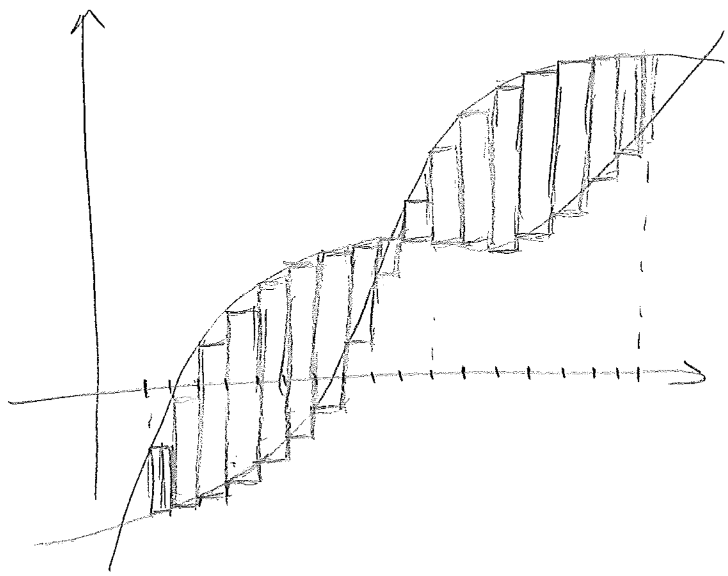
$$= 12u - u^3 \Big|_{-2}^{2} = 16 - (-16) = 32$$

BUT WHAT IF $f(x), g(x)$ INTERCEPT IN MULTIPLE POINTS AND/OR CHANGE SIGN?

GENERAL IDEA: DIVIDE AND CONQUER!

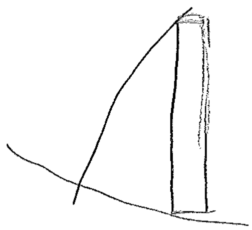


SUBDIVIDE $[a,b]$ IN n EQUAL PIECES



APPROXIMATE WITH RECTANGLES

WHAT'S THE HEIGHT OF A RECTANGLE?



$f(x_{i,n}^*)-g(x_{i,n}^*)$ WHICHEVER'S POSITIVE!

OR

$g(x_{i,n}^*)-f(x_{i,n}^*)$ SO IT'S $|f(x_{i,n}^*)-g(x_{i,n}^*)|$

WE GOT A RIEMANN SUM!

n-TH APPROX AREA $= \sum_{i=1}^{n} |f(x_{i,n}^*)-g(x_{i,n}^*)|$

TAKE THE LIMIT

$$\lim_{n\to\infty} \sum_{i=1}^{n} |f(x_{i,n}^* - g(x_{i,n}^*)| = \int_a^b |f(x)-g(x)\,dx \; !$$

DEF: AREA BETWEEN $f(x)$ AND $g(x)$ WITH x RUNNING FROM a TO b:

$$\int_a^b |f(x)-g(x)|\,dx$$

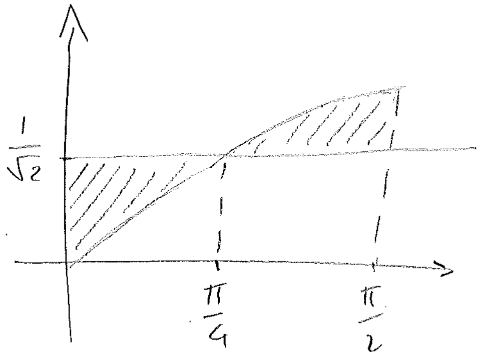
IN PRACTICE, TO COMPUTE IT WE'LL STILL HAVE TO DIVIDE AND CONQUER, I.E. FIND THE INTERSECTION POINTS.

EXAMPLE:

FIND THE AREA BETWEEN $y=\frac{1}{\sqrt{2}}$ AND $y=\sin(x)$ FOR x FROM 0 TO $\pi/2$.

SKETCH

$\sin(x)=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$ WHEN $x=\frac{\pi}{4}$



AREA A: x FROM 0 TO $\pi/4$

AREA B: x FROM $\frac{\pi}{4}$ TO $\pi/2$

TOTAL AREA = AREA A + AREA B =

$$\int_0^{\pi/4} \frac{1}{\sqrt{2}} - \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \frac{1}{\sqrt{2}} \, dx =$$

$$\left.\frac{x}{\sqrt{2}} + \cos x\right|_0^{\pi/4} + \left.\left(-\cos x - \frac{x}{\sqrt{2}}\right)\right|_{\pi/4}^{\frac{\pi}{2}} = \left(-1 + \frac{\pi}{4\sqrt{2}} + \frac{\sqrt{2}}{2}\right) +$$

$$\left(\frac{\pi}{4\sqrt{2}} + \frac{\sqrt{2}}{2} - \frac{\pi}{2\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}} - \frac{\pi}{2\sqrt{2}} - 1 + \frac{2\sqrt{2}}{2} = \sqrt{2} - 1$$