WARM-UP

① SUPPOSE WE KNOW THAT

$$\int f(x)\,dx = F(x) + C$$

USE THE F.T.C. TO FIND

$\int f(2x)\,dx$ IN TERM OF $F(x)$.

USE THIS TO SHOW THAT

$$\int_{a/2}^{b/2} f(2x)\,dx = \frac{1}{2}\int_a^b f(x)\,dx$$

② COMPUTE $\dfrac{d}{dx}\displaystyle\int_x^{x^3} e^{-\tau^2}\,d\tau$.

(SPLIT THE INTEGRAL AND THEN USE THE CHAIN RULE)

SOL:

① $(F(x))' = f(x)$ SO $(F(2x))' = 2f(x)$; THEN

$\frac{1}{2}F(2x)$ IS AN ANTIDER OF $f(2x)$. BY

FTC $\displaystyle\int_{a/2}^{b/2} f(2x)\,dx = \frac{F(2x)}{2}\Big|_{a/2}^{b/2} = \frac{1}{2}(F(b) - F(a))$

② $E(x) = \displaystyle\int_0^x e^{-\tau^2}\,d\tau$ THEN $\displaystyle\int_x^{x^3} e^{-\tau^2}\,d\tau = E(x^3) - E(x)$

$$(E(x^3) - E(x))' = 3x^2 E'(x^3) - E'(x) = 3x^2 e^{-x^6} - e^{-x^2}$$

SUBSTITUTION

WE NOW KNOW WHAT TO LOOK FOR WHEN COMPUTING INTEGRALS; TO COMPUTE

$$\int_a^b f(x)\,dx$$ WE PICK AN ANTIDERIVATIVE

$F(x)$ OF $f(x)$ AND THEN COMPUTE

$$F(b) - F(a)$$

NOTATION:

WE WRITE $F(x)\Big|_a^b$ FOR $F(b) - F(a)$

SAME FOR $\int f(x)\Big|_a^b$.

WE WILL ALSO WRITE $F(u)\Big|_{u=u(x)}$ FOR $F(u(x))$.

AT THE MOMENT WE HAVE A (RATHER SHORT) LIST OF FUNCTIONS FOR WHICH WE KNOW AN ANTIDERIVATIVE. WE CAN ALSO DEAL WITH LINEAR COMBINATIONS

$$\int Af(x) + Bg(x)\,dx = A\int f(x)\,dx + B\int g(x)\,dx$$

BUT HOW ABOUT PRODUCTS AND COMPOSITIONS OF FUNCTIONS IN OUR LIST?

THERE IS NO EASY PROCEDURE, CONTRARY TO WHAT HAPPENS WITH DERIVATIVES.
EVERY INTEGRAL IS A RIDDLE WE HAVE TO SOLVE.

SO WHAT CAN WE DO? OUR MAIN IDEA IS TO "TINKER" OR "MASSAGE" OUR FUNCTION $f(x)$ UNTIL IT BECOMES SOMETHING MANAGEABLE.

EXAMPLE:

WE CAN INTEGRATE $f(x)=e^x$, $g(x)=-x^2$, BUT NOT $f(g(x))=e^{-x^2}$

EXAMPLE:

ONE TYPE OF COMPOSITION WE CAN DEAL WITH IS WHEN $g(x)=Ax+B$

$$\int_a^b f(Ax+B) = \frac{F(Ax+B)}{A}\Bigg|_a^b$$

$$\int_0^1 \log(3x+1) = \frac{(3x+1)(\log(3x+1)-1)}{3}\Bigg|_0^1 =$$

$$\frac{4(\log 4-1)}{3}+\frac{1}{3}$$

ONE OF OUR BEST TOOLS TO SOLVE MORE GENERAL INTEGRALS IS THE CHAIN RULE

$$- \quad \frac{d}{dx}F(u(x)) = F'(u(x))\cdot u'(x)$$

WHAT DOES THIS TELL US? WELL, BY THE FTC IF $F'(x)=f(x)$ THEN:

SUBSTITUTION RULE:

$$\int f(u(x))\, u'(x)\, dx = F(u)\Big|_{u=u(x)}$$

SO HOW DO WE APPLY IT?

EXAMPLES:

- $\int x^3 e^{x^4} dx$ WE START FROM THE MORE COMPLICATED ONE

FACTORS

$e^{x^4} = e^{(u(x))}$ $\quad u(x) = x^4$

$u'(x) = 4x^3$ LOOKS LIKE OTHER FACTOR!

$$\int x^3 e^{x^4} dx = \frac{1}{4}\int 4x^3 e^{x^4} dx = \frac{1}{4}\int u' e^{u} du \Big|_{u=x^4}$$

$$= \frac{1}{4} e^{u}\Big|_{u=x^4} = \frac{e^{x^4}}{4}$$

- $\int 9 \sin^8(x)\cos(x)\, dx$ WE START BY GUESSING WHICH FACTOR SHOULD BE $u'(x)$;

EASY PICK IS $\cos(x)$! $\int \cos(x)\, dx = \sin(x) + C$

SO IF WE WRITE $u(x) = \sin(x)$ THEN

$$\int 9 \sin^8(x)\cos(x)\, dx = \int 9\, u(x)^8 \cdot u'(x)\, dx =$$

$$= \int 9 u^8\, du \Big|_{u=\sin(x)} = 9\frac{u^9}{9} + C \Big|_{u=\sin x}$$

$$= \sin(x)^9 + C$$

WHEN WE APPLY THE SUBTITUTION RULE TO DEFINITE INTEGRALS WE NEED TO BE A BIT MORE CAREFUL

SUBS. RULE, DEFINITE VERSION

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

SO AFTER WE APPLY SUBSTITUTION THE VARIABLE u TAKES ON ALL VALUES FROM $u(a)$ TO $u(b)$.

TWO METHODS TO SOLVE SAME INTEGRAL:

- USE SUBSTITUTION TO FIND
  $$\int f(u(x))\,u'(x)\,dx \quad \text{(FUNCTION OF } x\text{)}$$
  AND EVALUATE AT $x=a$, $x=b$

- FIND
  $$\int f(u)\,du \quad \text{(FUNCTION OF } u\text{)}$$
  AND EVALUATE AT $u=u(a)$ $u=u(b)$

THE SECOND OPTION IS OFTEN EASIER FOR DEFINITE INTEGRALS

EXAMPLE:

- $$\int_0^1 x^2 \sin(x^3+1)\,dx$$ WE TRY PICKING "$u'(x) = x^2$"

FACTORS

NOW, $\sin(x^3+1) = \sin(u)$ WITH $u = x^3+1$;

$u' = 3x^2$ WHICH IS "ALMOST" x^2; SO WITH SOME TINKERING

$$\int_0^1 x^2 \sin(x^3+1)\,dx = \frac{1}{3}\int_0^1 3x^2 \sin(x^3+1)\,dx$$

$$= \frac{1}{3}\int_0^1 u'(x)\sin(u(x))\,dx = \frac{1}{3}\int_{u(0)}^{u(1)} \sin(u)\,du$$

$$= \frac{1}{3}\int_1^2 \sin(u)\,du = \frac{1}{3}\left(-\cos(u)\right)\Big|_1^2 = \frac{1}{3}\cdot\left(\cos(1)-\cos(2)\right)$$

RECALL $u = x^3+1$

- $$\int_0^1 \frac{x}{1+x^2}\,dx$$ WE REWRITE THE INTEGRAND AS $x \cdot \frac{1}{1+x^2}$

IF $u = 1+x^2$ THEN $u' = 2x$ AND

$$x \cdot \frac{1}{1+x^2} = \frac{1}{2}\cdot u' \cdot \frac{1}{u}$$ SO

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{u=1}^{u=2} \frac{1}{u}\,du = \frac{\log(u)}{2}\Big|_1^2 = \frac{\log 2}{2}$$