DEF: AREA BETWEEN $f(x)$ AND $g(x)$ WITH x RUNNING FROM a TO b:

$$\int_a^b |f(x)-g(x)|\,dx$$

IN PRACTICE, TO COMPUTE IT WE'LL STILL HAVE TO DIVIDE AND CONQUER, I.E. FIND THE INTERSECTION POINTS.

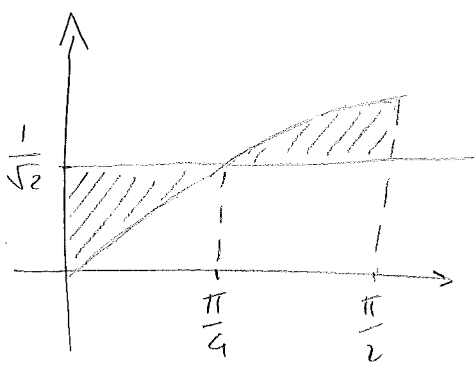
EXAMPLE:
FIND THE AREA BETWEEN $y=\frac{1}{\sqrt{2}}$ AND $y=\sin(x)$ FOR x FROM 0 TO $\pi/2$.

SKETCH

$\sin(x)=\frac{1}{\sqrt{2}}$

$=\frac{\sqrt{2}}{2}$ WHEN

$x=\frac{\pi}{4}$



AREA A:
x FROM 0 TO $\pi/4$

AREA B:
x FROM $\frac{\pi}{4}$ TO $\frac{\pi}{2}$

TOTAL AREA = AREA A + AREA B =

$$\int_0^{\pi/4} \frac{1}{\sqrt{2}} - \sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \frac{1}{\sqrt{2}}\,dx =$$

$$\frac{x}{\sqrt{2}} + \cos x\Big|_0^{\pi/4} + \left(-\cos x - \frac{x}{\sqrt{2}}\right)\Big|_{\pi/4}^{\frac{\pi}{2}} = \left(-1+\frac{\pi}{4\sqrt{2}}+\frac{\sqrt{2}}{2}\right)+$$

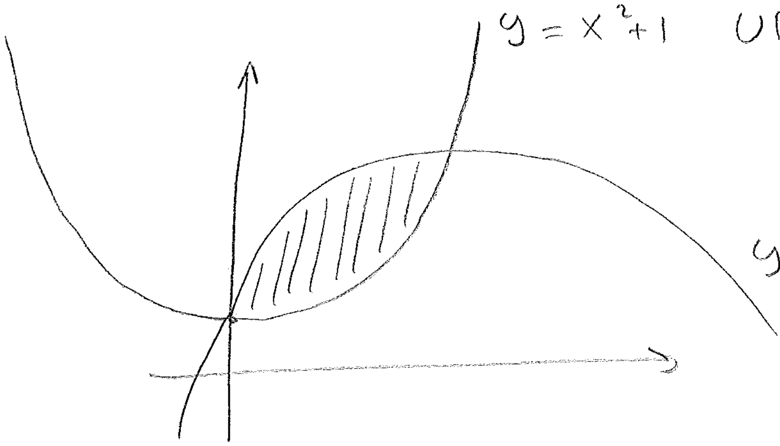
$$\left(\frac{\pi}{4\sqrt{2}}+\frac{\sqrt{2}}{2}-\frac{\pi}{2\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}} - \frac{\pi}{2\sqrt{2}} - 1 + \frac{2\sqrt{2}}{2} = \sqrt{2}-1$$

EXAMPLE:

FIND THE AREA OF THE REGION ENCLOSED BY

$$y = \underbrace{x^2+1}_{T(x)} \quad \text{AND} \quad y = \underbrace{6x-3x^2+1}_{B(x)}$$

STEP 1: SKETCH



$y = x^2+1$ UPWARD FACING PARABOLA

$y = 6x-3x^2+1$ DOWNWARD FACING PAR.

STEP 2: FIND INTERSECTIONS $T(x) = B(x) \sim$

$$x^2+1 = 6x-3x^2+1 \sim 4x^2-6x = 0$$

$$\sim x(4x-6) = 0 \qquad x = 0, \tfrac{3}{2}$$

STEP 3: COMPUTE INTEGRAL (KNOWING THAT $B(x) \geq T(x)$ ON OUR INTERVAL)

$$\int_0^{\frac{3}{2}} B(x) - T(x)\, dx = \int_0^{\frac{3}{2}} 6x - 4x^2\, dx = 3x^2 - \frac{4}{3}x^3 \Big|_0^{\frac{3}{2}}$$

$$= 3 \cdot \frac{9}{4} - \frac{4}{3} \cdot \frac{27}{8} = \frac{27}{4} - \frac{18}{4} = \frac{9}{4}$$

ONE MORE INTRICATE EXAMPLE:

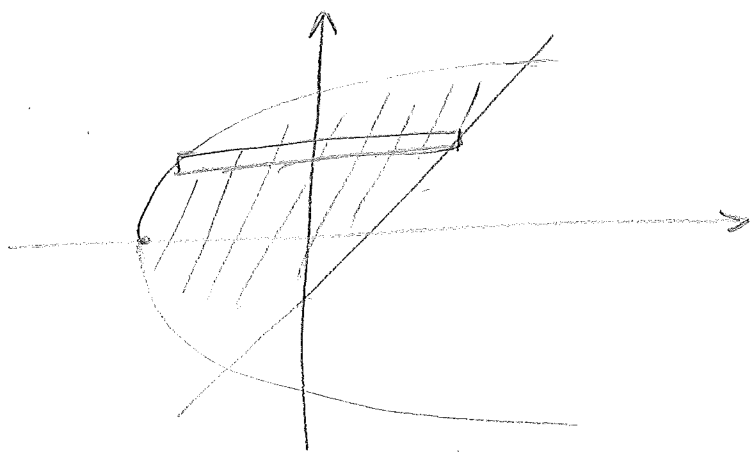
CURVES DO NOT NEED TO BE IN THE FORM $y = f(x)$!

EXAMPLE:

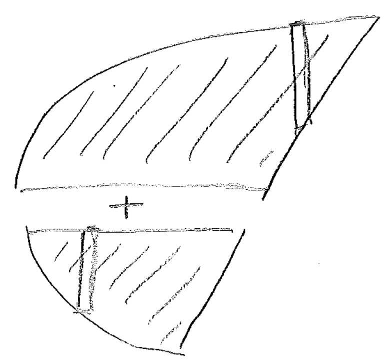
FIND THE AREA ENCLOSED BY

$y^2 = 2x + 6$ AND $y = x - 1$

SKETCH: $y^2 = 2x + 6 \sim x = -3 + \frac{y^2}{2}$, A PARABOLA



APPROACH 1:
SPLIT AS



APPROACH 2: y AS A VARIABLE!

$x = \underbrace{\frac{y^2}{2} - 3}_{T(y)}$ $\quad x = \underbrace{y + 1}_{B(y)}$

$T(y) = B(y) \sim \frac{y^2}{2} - 3 = y + 1 \sim y^2 - 2y - 8 = 0$

$y = -2, 4$

$$\int_{-2}^{4} B(y) - T(y)\,dy = \int_{-2}^{4} -\frac{y^2}{2} + y + 4\,dy = -\frac{y^3}{6} + \frac{y^2}{2} + 4y \Big|_{-2}^{4}$$

$$= \left(-\frac{64}{6} + \frac{16}{2} + 16\right) - \left(\frac{8}{6} + \frac{4}{2} - 8\right) = 18$$