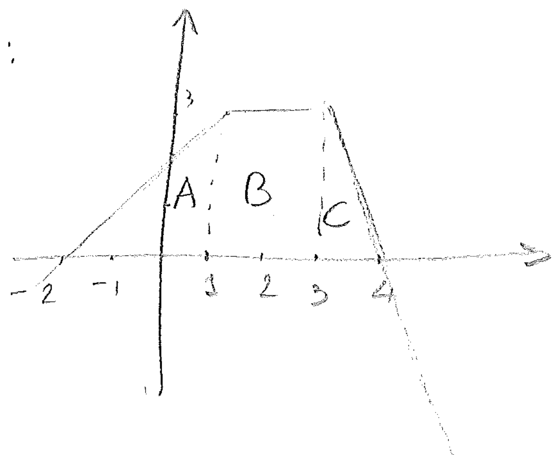
WARM UP:

(1) USING SOME GEOMETRY AND THE ARITHMETIC RULES FOR DOMAINS OF INTEGRATION, SOLVE THE FOLLOWING INTEGRAL:

$$\int_0^4 f(x)\,dx \qquad f(x) = \begin{cases} x+2 & x<1 \\ 3 & 1 \le x < 3 \\ 12-3x & x \ge 3 \end{cases}$$

SOL:



$$\int_0^4 f(x)\,dx = \underbrace{\int_0^1 f(x)\,dx}_{\text{"}A} + \underbrace{\int_1^3 f(x)\,dx}_{\text{"}B} + \underbrace{\int_3^4 f(x)}_{\text{"}C}$$

A: BIG $\triangle$ − SMALL $\triangle$ $\quad \frac{3\cdot 3}{2} - \frac{2\cdot 2}{2} = \frac{5}{2}$

B: RECTANGLE $\quad 2\cdot 3 = 6$

C: SQ. $\triangle$ $\quad \frac{3\cdot 1}{2} = \frac{3}{2}$

TOTAL $\quad \frac{5}{2} + 6 + \frac{3}{2} = 10$

$$\int_0^4 f(x)\,dx = 10$$

WARM-UP:

(2) FIND $a, b, f(x)$ SUCH THAT

$$\sum_{i=0}^{3} 1 + \frac{i}{4} \quad \text{IS A } \underline{\text{LEFT}} \text{ R.S.}$$

FOR $f(x)$ ON $[a, b]$ (MANY POSSIBLE SOLUTIONS)

SOL:

WE WANT

$$\sum_{i=0}^{3} 1 + \frac{i}{4} = \sum_{i=0}^{3} f(a + i\Delta_x) \cdot \Delta_x$$

WHERE $\Delta_x = \dfrac{b-a}{4}$ ← 4 SQUARES!

FIRST, WE MAY ASSUME $a=0$ SO

$$\sum_{i=0}^{3} 1 + \frac{i}{4} = \sum_{i=0}^{3} f\left(\frac{ib}{4}\right) \cdot \frac{b}{4}$$

WE LOOK FOR A LINEAR $f(x) = cx + d$

$$\sum_{i=0}^{3} 1 + \frac{i}{4} = \sum_{i=0}^{3} \left(c\left(\frac{ib}{4}\right) + d\right) \frac{b}{4} \qquad \frac{d \cdot b}{4} = 1$$

$\underline{i=0}$ $\quad 1 = \dfrac{d \cdot b}{4}$ $\qquad \underline{i=3}$ $\quad \dfrac{3}{16} c b^2 + 1 = \dfrac{7}{4}$ ~

$\dfrac{3}{16} c b^2 = \dfrac{3}{4}$ ~ $c \cdot b^2 = 4$ $\qquad$ WE CAN PICK b;

FOR $b = 1$ $\; c = 4, d = 4$, $\underline{f(x) = 4x + 4 \text{ ON } [0, 1]}$

ANOTHER WAY OF SEEING AN INTEGRAL

A PARTICLE IS MOVING ALONG THE y AXIS WITH VELOCITY $v(T)$. CALL ITS HEIGHT $h(T)$. CAN WE APPROXIMATE HOW MUCH THE PARTICLE MOVES FROM A TIME $T=a$ TO $T=b>a$?

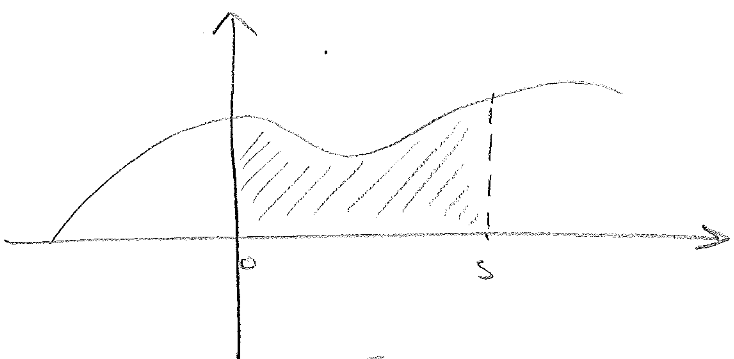
- IF WE DIVIDE THE INTERVA $[a,b]$ IN VERY SMALL SUBINTERVALS $[x_i, x_{i+1}]$ WE MAY SUPPOSE $v(T)$ IS "ALMOST" CONSTANT ON $[x_i, x_{i+1}]$
- SAY $v(T)$ IS ABOUT v_i IN $[x_i, x_{i+1}]$; THE IN THE TIME FROM $T=x_i$ TO $T=x_{i+1}$ THE PARTICLE MOVES BY $(x_{i+1}-x_i)\cdot v_i$
- SO THE TOTAL MOTION IS $\sum_{i=1}^{m} v_i(x_{i+1}-x_i)$
- BUT! WE CAN PICK THE x_i SO THAT THE SEGMENTS ARE EQUAL $x_{i+1}-x_i = \frac{b-a}{m}$! AND WE CAN PICK $v_i = v(x^*_{i,m})$!
- THEN OUR APPROXIMATION IS $h(b)-h(a) \cong \sum_{i=1}^{m} v(x^*_{i,m})\Delta x$
- TAKING THE LIMIT WE GET

$$h(b)-h(a) = \int_a^b v(T)\,dT \;!!$$

"THE DERIVATIVE" OF THE INTEGRAL

IF $\int_a^b f(x)\,dx$ IS THE "SIGNED" AREA BELOW THE CURVE $(x, f(x))$ THE WE CAN CONSIDER SOMETHING SUCH AS

$$F(s) = \int_0^s f(x)\,dx$$ WHERE s IS A VARIABLE; IT REPRESENTS THE SIGNED AREA BELOW THE CURVE BETWEEN 0 AND s

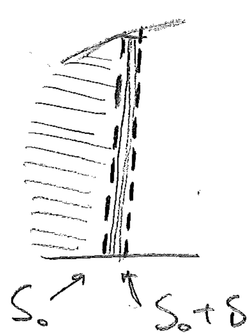


WHAT DO WE KNOW ABOUT $F(s)$? IS IT DIFFERENTIABLE?

TO ANSWER THIS, PICK A POINT s_0. WE ARE LOOKING FOR SOME α S.T.

$$F(s_0+\delta) \approx F(s_0) + \alpha \cdot \delta$$ FOR SMALL VALUES OF δ

ZOOM IN:



IF δ IS SMALL ENOUGH WE CAN THINK WE ARE JUST ADDING <u>ONE</u> RECTANGLE TO OUR (LEFT) RIEMANN SUM! WHAT'S THE AREA? $\delta \cdot f(s_0)$!

SO INTUITIVELY, $\frac{d}{ds} F(s)\big|_{s_0} = f(s_0)$!!

$$\left(\text{"}F'(s_0)\text{"} \right)$$

NOTE:

THE FIRST EXAMPLE JUSTIFIES WHY WE TAKE SIGNED AREAS (IF $v(\tau)<0$ WE'RE MOVING BACKWARDS) AND WHY IF WE HAVE $b<a$ THE INTEGRAL CHANGES SIGN (WE ARE "GOING BACKWARDS IN TIME")

THESE -HOPEFULLY ILLUMINATING- EXAMPLES LEAD US TO WHAT IS CALLED, AND FOR GOOD REASON, THE

FUNDAMENTAL THEOREM OF CALCULUS:

• LET $a<b$ AND LET $f(x)$ BE A CONTINUOUS FUNCTION ON $[a,b]$

PART 1) LET $F(x)=\int_a^x f(\tau)\,d\tau$ FOR $x\in[a,b]$

THE FUNCTION $F(x)$ IS DIFFERENTIABLE AND $F'(x)=f(x)$

PART 2) LET $G(x)$ A DIFFERENTIABLE FUNCTION ON $[a,b]$ WITH $G'(x)=f(x)$ FOR ALL $a<x<b$. THEN

$$\int_a^b f(x)\,dx = G(b)-G(a)$$