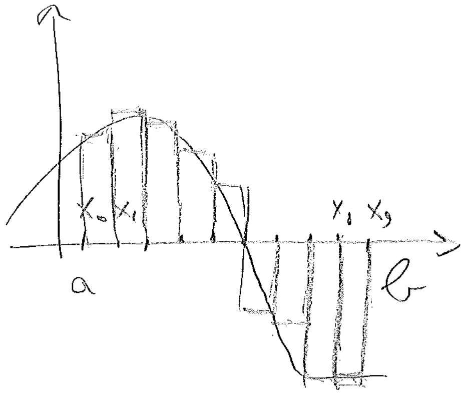
BACK TO THE DEFINITION OF INTEGRAL



$f(x)$ DEFINED ON $[a,b]$

- $[a,b]$ IS PARTITIONED IN n EQUAL INTERVALS OF WIDHT $x_{i+1} - x_i = \frac{b-a}{n} = \Delta_x$

- ON EACH INTERVAL WE PICK A RECTANGLE WITH "SIGNED HEIGTH" $f(x^*_{i,n})$

$x^*_{i,n}$

$*$ ← JUST TO REMIND US IT CAN BE ANY POINT WITHIN OUR INTERVAL

n ← WE SPLIT $[a,b]$ IN n PARTS

i ← IN THE INTERVAL $[x_i, x_{i+1}]$

- WE CALL THE SUMMATION

$$\sum_{i=1}^{n} f(x^*_{i,n}) \frac{b-a}{n} = \sum_{i=1}^{n} f(x^*_{i,n}) \Delta_x$$

A <u>RIEMANN SUM</u>

AN EQUIVALENT WAY TO DEFINE $\int_a^b f(x)\,dx$ IS:

SUPPOSE THAT AS n INCREASES, <u>ALL</u> RIEMANN SUMS CONVERGE TO THE <u>SAME</u> NUMBER L.

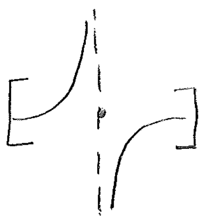
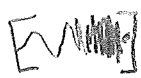
THEN $\int_a^b f(x)\,dx = L$

IF f ONLY HAS A <u>FINITE AMOUNT</u> OF <u>JUMP</u> DISCONTINUITIES ON $[a,b]$ THA ABOVE <u>ALWAYS</u> <u>HAPPENS</u>

JUMP DISCONTINUITIES:

 ✓  ✓  ✓

$$f(x) = \begin{cases} 1 & x < \sqrt{2} \\ 0 & x \geq \sqrt{2} \end{cases} \quad ✓ \quad \text{ON } [0, 2]$$

 ×  ×

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \times \quad \text{ON } [-1, 1]$$

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \times \quad \text{ON } [-1, 1]$$

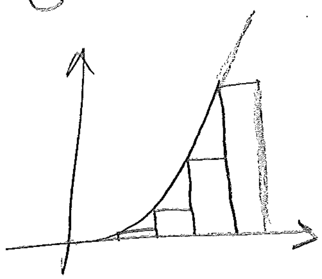
<u>NOTE</u>: A FUNCTION CAN HAVE WORSE DISCONUITIES AND STILL BE INTEGRABLE, BUT WE NEED TO CHECK CASE BY CASE.

NOW SUPPOSE OUR $f(x)$ IS "DECENT", THAT IS, ONLY F.M. JUMP DISCONTINUITIES ON $[a, b]$.
THEN <u>ANY</u> SEQUENCE OF RIEMANN SUMS WILL CONVERGE TO $\int_a^b f(x)\,dx$ <u>AS LONG AS</u> $m \to \infty$. SO WE CAN PICK OUR FAVOURITE ONE!

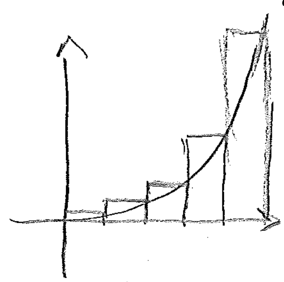
- LEFT R.S.: $x_{i,m}^* = x_i$
- RIGHT R.S.: $x_{i,m}^* = x_{i+1}$
- MIDDLE R.S.: $x_{i,m}^* = \dfrac{x_i + x_{i+1}}{2} = x_i + \dfrac{\Delta x}{2}$

EXAMPLES

$f(x) = x^2 \qquad [a,b] = [0,1]$



LEFT



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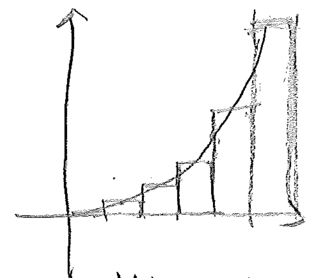


MIDDLE

$$\sum_{i=0}^{m} \left(\frac{i}{m}\right)^2 \cdot \frac{1}{m} \qquad \sum_{i=0}^{m} \left(\frac{i+1}{m}\right)^2 \cdot \frac{1}{m} \qquad \sum_{i=0}^{m} \left(\frac{i+\frac{1}{2}}{m}\right) \cdot \frac{1}{m}$$

LET'S TRY COMPUTING THE LIMIT:

$$\lim_{m\to\infty} \sum_{i=0}^{m} \left(\frac{i}{m}\right)^2 \cdot \frac{1}{m} = \lim_{m\to\infty} \sum_{i=0}^{m} \frac{i^2}{m^3} = \lim_{m\to\infty} \frac{1}{m^3} \sum_{i=0}^{m} i^2 =$$

WE PICK THE LEFT SUM

$$= \lim_{m\to\infty} \frac{m(m+1)(2m+1)}{6m^3} = \frac{2}{6} = \frac{1}{3}$$

SEEN LAST CLASS

QUESTION:

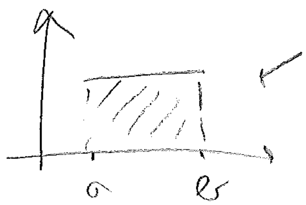
LET $f(x) = 2x - 3 \qquad [a,b] = [0,2]$

- COMPUTE THE LEFT, RIGHT AND MIDDLE R.S. FOR $m = 2, 4$
- FOR A GIVEN m, HOW DO THEY COMPARE, WHICH IS BIGGER?

SO NOW WE HAVE (AND HOPEFULLY UNDERSTAND) THE DEFINITION, BUT IT INVOLVES LIMITS. LIMITS ARE HARD!

CAN WE COMPUTE SOME $\int_a^b f(x)\,dx$
ONLY KNOWING THIS, NO LIMITS INVOLVED?
WE CAN SOLVE THE LIMIT, BUT FOR NOW LET'S USE SOME <u>GEOMETRY</u>:

$\int_a^b c\,dx$ $\quad c$ CONSTANT

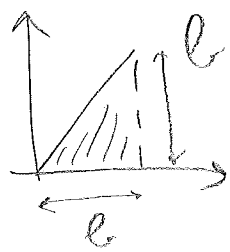


RECTANGLE!
AREA $= (b-a)\cdot c$

IF $c<0$? $(b-a)\cdot c$ AGAIN

$\int_0^b x\,dx = \frac{b^2}{2}$

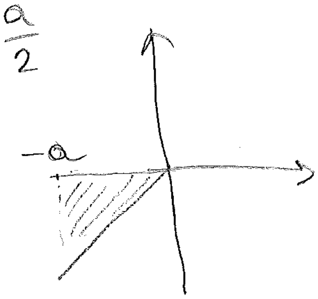
$a=0$



SQUARE TRIANGLE!
AREA $b\cdot\frac{b}{2}$

$\int_{-a}^0 x\,dx = -\frac{a^2}{2}$

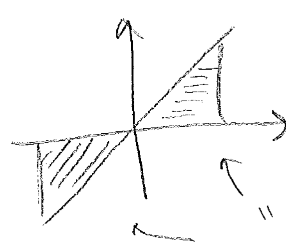
$b=0$



SAME!
AREA $a\cdot\frac{a}{2}$

(EXERCISE: COMPUTE $\int_0^b cx\,dx$)

$\int_{-a}^a x\,dx = 0$



"NEGATIVE" AREA AS BIG AS "POSITIVE" AREA

(EXERCISE: COMPUTE $\int_{-a}^a \sin x\,dx$)

ARITHMETIC OF INTEGRATION

SUMMATIONS AND LIMITS BOTH SATISFY BASIC ARITHMETIC RULES. IT STANDS TO REASON THAT INTEGRALS SHOULD DO THE SAME:

THM (ARITHMETIC OF INTEGRATION):

LET f, g BE DEFINED ON $[a,b]$ AND SUPPOSE $\int_a^b f(x)\,dx$ AND $\int_a^b g(x)\,dx$ EXIST. THEN

- $\int_a^b f(x) + g(x)\,dx \underset{*}{=} \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b c\,f(x) \underset{*}{=} c \int_a^b f(x)$ FOR A CONSTANT c
- $\int_a^b f(x) - g(x)\,dx = \int_a^b f(x) - \int_a^b g(x)$

MOREOVER

- $\int_a^b 1\,dx = b - a$

NOW CONSIDER THIS. WHAT IF WE HAVE $b < a$? DOES THE DEFINITION OF INTEGRAL STILL MAKE SENSE? YES! WE'LL JUST HAVE A NEGATIVE Δ_x

* WE ARE ALSO IMPLICITY STATING THAT IF THE LHS EXISTS SO DOES THE RHS

REMARK:

THE DEFINITION OF $\int_a^b f(x)\,dx$ AND EVERYTHING WE DID EXTENDS TO THE CASE $b<a$

WE CAN NOW DO SOME "ARITHMETIC" ON THE INTERVAL $[a,b]$ TOO

THM:

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- IF $a<b<c$ THEN $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$

EXERCISE:

LET $f(x)$ BE DEFINED ON $[-a,a]$ FOR SOME $a>0$. SHOW THAT:

- IF $f(x)=f(-x)$ THEN $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$
- IF $f(x)=-f(-x)$ THEN $\int_{-a}^a f(x)\,dx = 0$

ANOTHER WAY OF SEEING AN INTEGRAL

A PARTICLE IS MOVING ALONG THE y AXIS WITH VELOCITY $v(T)$. CALL ITS HEIGHT $h(T)$. CAN WE APPROXIMATE HOW MUCH THE PARTICLE MOVES FROM A TIME $T=a$ TO $T=b>a$?

- IF WE DIVIDE THE INTERVA $[a,b]$ IN VERY SMALL SUBINTERVALS $[x_i, x_{i+1}]$ WE MAY SUPPOSE $v(T)$ IS "ALMOST" CONSTANT ON $[x_i, x_{i+1}]$
- SAY $v(T)$ IS ABOUT v_i IN $[x_i, x_{i+1}]$; THE IN THE TIME FROM $T=x_i$ TO $T=x_{i+1}$ THE PARTICLE MOVES BY $(x_{i+1}-x_i)\cdot v_i$
- SO THE TOTAL MOTION IS $\sum_{i=1}^{m} v_i(x_{i+1}-x_i)$
- BUT! WE CAN PICK THE x_i SO THAT THE SEGMENTS ARE EQUAL $x_{i+1}-x_i = \frac{b-a}{m}$! AND WE CAN PICK $v_i = v(x^*_{i,m})$!
- THEN OUR APPROXIMATION IS $h(b)-h(a) \cong \sum_{i=1}^{m} v(x^*_{i,m})\Delta x$
- TAKING THE LIMIT WE GET

$$h(b)-h(a) = \int_a^b v(T)\,dT \quad !!$$