WARM-UP

① SUPPOSE $\int_a^b f(x)\,dx = 30$.

WHAT IS $\int_{a/2}^{b/2} f(2x)\,dx$?

② WRITE $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\log\left(3+\frac{5i}{n}\right)\cdot 5}{n}$

AS AN INTEGRAL.

IS IT A LIMIT OF RIGHT OR LEFT RIEMANN SUMS?

Sol:

① THE FUNCTION $f(2x)$ TAKES THE SAME VALUES ON $[a/2, b/2]$ AS $f(x)$ ON $[a,b]$.

IF WE WRITE DOWN A LEFT R.S. FOR BOTH WE SEE THAT THE HEIGHTS OF RECT. ARE THE SAME, BUT THE BASE HALF.

SO $\int_{a/2}^{b/2} f(2x)\,dx = \frac{1}{2}\int_a^b f(x)\,dx = 15$

② $\lim_{n\to\infty} \sum_{i=1}^{n} \log\left(\underset{3=a,\ 8=b}{3+\frac{5i}{n}}\right) \underbrace{\frac{5}{n}}_{\Delta x} = \int_3^8 \log(x)\,dx$

RIGHT SUM AS IT STARTS FROM $3+\Delta x$

LET'S SET UP SOME NOTATION

DEF: LET $f(x)$ BE A FUNCTION. A FUNCTION $F(x)$ IS AN ANTIDERIVATIVE OF $f(x)$ ON A GIVEN INTERVAL IF $F'(x) = f(x)$ ON SAID INTERVAL.

THE F.T.C. SHOWS THAT ANY CONTINUOUS FUNCTION HAS AN ANTIDERIVATIVE. WHAT IF WE HAVE TWO ANTIDERIVATIVES FOR $f(x)$, SAY $G(x)$ AND $F(x)$?

$$\frac{d}{dx}\left(G(x) - F(x)\right) = f(x) - f(x) = 0$$

SO $G(x) - F(x)$ IS A CONSTANT!

PROP: GIVEN ANY ANTIDER. $F(x)$ OF $f(x)$, ANY OTHER ANTIDER. OF $f(x)$ IS IN THE FORM

$$F(x) + C$$ WHERE C IS A NUMBER

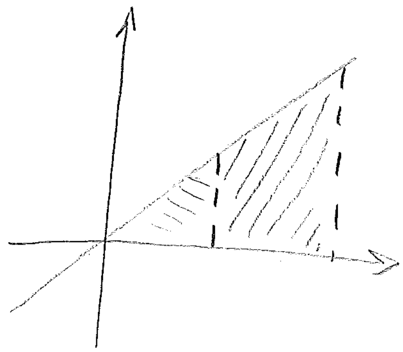
DEF: THE INDEFINITE INTEGRAL OF, DENOTED $\int f(x)\,dx$ IS $F(x) + C$ WHERE $F(x)$ IS AN ANTIDER. OF $f(x)$ AND C IS AN ARBITRARY CONSTANT.

EXAMPLES:

- LET'S CHECK THE THM IN AN EASY CASE

WE SHOWED THAT $\int_0^b x\,dx = \frac{b^2}{2}$

IN GENERAL $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$



(BIG TRIANG \ SMALL TRIANG)

IT'S EASY TO SEE THAT IF $G(x) = \frac{x^2}{2}$

$G'(x) = x$

BY PART 2 $\int_a^b x\,dx = G(b) - G(a) = \frac{b^2}{2} - \frac{a^2}{2}$

CONVERSELY, USING THE EQUALITY ABOVE $F(x) = \int_0^x \tau\,d\tau = \frac{x^2}{2}$ (RECALL THAT $\int_0^x x\,dx$ IS GIBBERISH)

AND $F'(x) = x$ AS PREDICTED BY PART 1

- WE WANT $\frac{d}{dx}\int_0^x e^{-\tau^2}\,d\tau$. WHILE EARLIER WE HAD TWO WAYS TO SOLVE THE PROBLEM, EITHER APPLY PART 1 OR FIND $F(x)$ AND THEN TAKE THE DERIVATIVE.

WE CANNOT DO THIS IN THIS CASE!
INDEED IT IS A (DIFFICULT) THEOREM
THAT THE FUNCTION $\int_0^x e^{-\tau^2} d\tau$ CANNOT
BE WRITTEN EXPLICITLY.

BUT OF COURSE WE CAN STILL APPLY
PART 1 WITH $f(x) = e^{-x^2}$ TO GET

$$\frac{d}{dx} \int_0^x e^{-\tau^2} d\tau = e^{-x^2}$$

- WHAT IF WE COMPLICATE THE INTERVALS OF INTEGRATION? WE WANT

$$\frac{d}{dx} \int_0^{x^2} e^{-\tau^2} d\tau$$

IF WE SET $E(x) = \int_0^x e^{-\tau^2} d\tau$

THEN $\int_0^{x^2} e^{-\tau^2} d\tau = E(x^2)$

THEN $\frac{d}{dx} E(x^2) \underset{\uparrow}{=} 2x\, E'(x^2)$

CHAIN RULE! $(f(g(x)))' = g'(x) f'(g(x))$

NOW BY PART 1 $E'(x) = e^{-x^2}$ SO

$$\frac{d}{dx} \int_0^{x^2} e^{-\tau^2} d\tau = 2x\, e^{-x^4}$$

EXAMPLES:

- FIND AN ANTIDERIVATIVE OF THE FOLLOWING FUNCTIONS:
  - x^3
  - x^m
  - e^{2x}
  - $\sin(x)$
  - $\frac{1}{x}$
  - $\log(x)$

SOL:
WE'LL LOOK FOR DERIVATIVES WE KNOW TO FIND SOMETHING SIMILAR, THEN ADJUST IT

- WE KNOW THAT $\frac{d}{dx} x^4 = 4x^3$; SO $F(x) = \frac{x^4}{4}$
- $\frac{d}{dx} x^{m+1} = m+1\, x^m$, SO $F(x) = \frac{x^{m+1}}{m+1}$
- $\frac{d e^{2x}}{dx} = 2e^{2x}$ SO $F(x) = \frac{e^{2x}}{2}$
- $\frac{d}{dx} \cos x = -\sin x$ SO $F(x) = -\cos(x)$
- $\frac{d}{dx} \log(x) = \frac{1}{x}$ FOR $x > 0$ $\frac{d}{dx} \log(-x) = \frac{1}{x}$ FOR $x < 0$
  SO $F(x) = \log(|x|)$

- $\frac{d}{dx} x \log x = \log x + 1$ WE WANT

TO REMOVE THE $+1$, SO WE ADD $-x$

$\frac{d}{dx} x \log x - x = \log x$, so $F(x) = x(\log x - 1)$