SUMMATION

TWO NOTATIONS (AND A HALF) FOR SUMMATIONS:

LET $m \leq n$ BE INTEGERS

IF $f(x)$ IS A FUNCTION DEFINED ON INTEGERS, WE WRITE

- $\sum_{k=m}^{n} f(k)$ TO MEAN THE SUM OF THE VALUES $f(m), f(m+1), \ldots, f(n)$, I.E.
- $$\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + \ldots + f(n-1) + f(n)$$

IF $a_m, a_{m+1}, \ldots, a_n$ ARE REAL NUMBERS WE WRITE

- $\sum_{i=m}^{n} a_i$ TO MEAN THE SUM OF THE a_i, I.E.
- $$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

EXAMPLE:

$$\sum_{k=2}^{5} \frac{1}{k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$$

NOTE: THE VARIABLE K IS WHAT WE CALL A "DUMMY" VARIABLE.

- EXPRESSION EVALUATES TO SOMETHING NOT CONTAINING K
- K ONLY "EXISTS" FOR THE SAKE OF OUR COMPUTATION AND COULD HAVE ANY NAME
- E.G. $k \sum_{i=1}^{10} \frac{1}{k^3}$ IS GIBBERISH

- MANY DIFFERENT EXPRESSION CAN HAVE THE SAME VALUE

$$\sum_{m=1}^{10} \frac{1}{\ln(m+1)} \quad \text{IS THE SAME AS} \quad \sum_{m=2}^{11} \frac{1}{\ln(m)}$$

$$\text{AND} \quad \frac{1}{\ln(2)} + \dots + \frac{1}{\ln(11)}$$

SUMMATIONS EVALUATE TO NUMBERS, SO IT MAKES SENSE THAT THEY HAVE ARITHMETIC RULES

THM:

- $\sum_{i=m}^{n} c \cdot a_i = c\left(\sum_{i=m}^{n} a_i\right)$ FOR A NUMBER c
- $\sum_{i=m}^{n} a_i + b_i = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

- NOTE: HARDER FOR <u>INFINITE SUMS</u>

<u>EXERCISE</u>:

USE THE TWO POINTS ABOVE TO SHOW THAT

$$\sum_{i=m}^{n} a_i - b_i = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

IT'S RARE THAT GIVEN A FUNCTION WE CAN WRITE $\sum_{i=0}^{m} f(i)$ EXPLICITLY. THE FOLLOWING TWO CASES ARE IMPORTANT:

- $\sum_{i=0}^{m} r^i = \frac{r^{i+1}-1}{r-1}$ FOR ALL NUMBERS $r \neq 1$
- $\sum_{i=0}^{m} i = \frac{i(i+1)}{2}$
- $\sum_{i=0}^{m} i^2 = \frac{1}{6} m(m+1)(2m+1)$

PROOF: FIRST: $(r-1)(r^i + \dots + 1) = r^{i+1} - 1$

SECOND: $S(x) = 1 + x + x^2 + \dots + x^m \text{ "=" } \frac{(x^{m+1}-1)}{x-1}$

$S'(x) = 1 + 2x + 3x^2 + \dots mx^{m-1} \text{ "=" } \left(\frac{(x^{m+1}-1)}{x-1}\right)'$

COMPUTE $S'(1)$

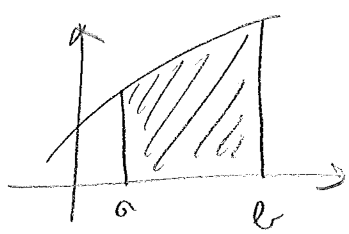
DEFINING THE <u>DEFINITE INTEGRAL</u> (AFTER RIEMANN)

NOTATION:

- $f(x)$ FUNCTION DEFINED ON $[a,b]$
- $\int_a^b f(x)\,dx$ "THE DEFINITE INTEGRAL OF $f(x)$ FROM a TO b"
- a, b "LIMITS OF INTEGRATION"; $[a,b] = a \le x \le b$ "INTERVAL OF INTEGRATION"

WHAT DO WE WANT $\int_a^b f(x)\,dx$ TO BE?

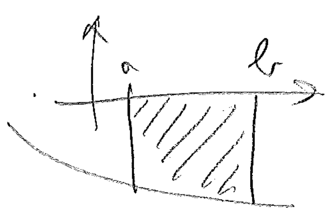
- IF $f(x)$ IS POSITIVE ON $[a,b]$



THEN $\int_a^b f(x)\,dx$ SHOULD BE THE AREA OF

$$\underline{\{(x,y) \mid a \le x \le b,\ y \le f(x)\}}$$

- WE WILL THINK OF "SIGNED" AREAS, SO IF $f(x)$ IS NEGATIVE WE'LL GET NEGATIVE



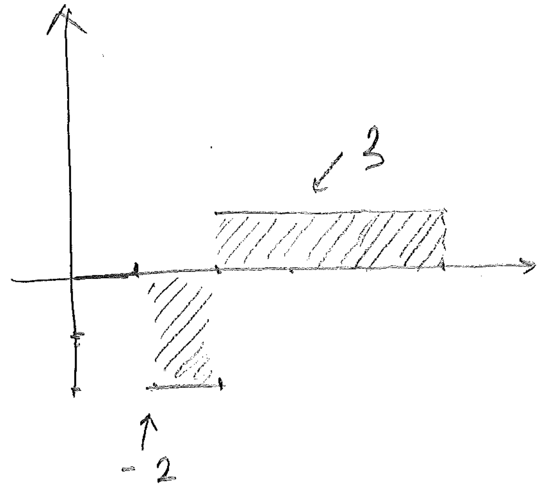
THE AREA OF

$$\{(x,y) \mid a \le x \le b,\ y \ge f(x)\}$$

- IF $f(x)$ HAS BOTH NEGATIVE AND POSITIVE VALS WE'LL GET BOTH

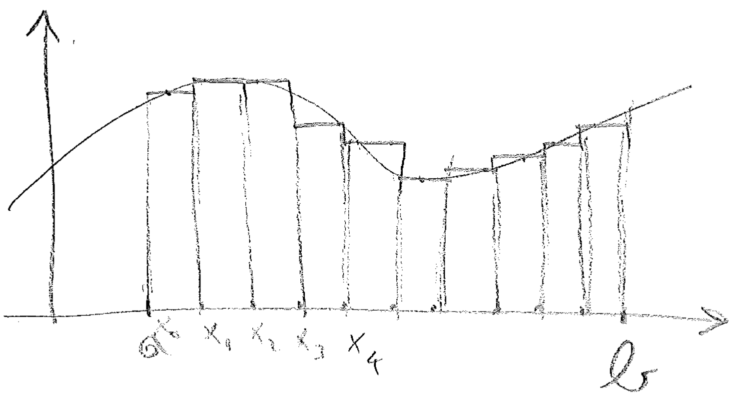
$$f(x) = \begin{cases} -2 & \text{IF } 1 \le x \le 2 \\ 1 & \text{IF } 2 < x \le 5 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\int_0^6 f(x)\,dx = -2 + 3 = 1$$



- WOULD THE NOTATION $\int_a^b f(x)\,dx$ MAKE SENSE IF $b < a$? YES, WE'LL SEE LATER

SO HOW DO WE ACTUALLY <u>DEFINE</u> IT? BY <u>APPROXIMATION</u>



(1) SELECT A NUMBER m AND DIVIDE $[a, b]$ IN m <u>EQUAL</u> INTERVALS

(2) $x_i = a + i \underbrace{\frac{b-a}{m}}_{=\Delta x}$, SUBINTERVALS $[x_i, x_{i+1}]$

(3) ON EACH SUBINT PICK AT RANDOM A NUMBER $x_i \le x^*_{i,m} \le x_{i+1}$ (EXAMPLE: MIDPOINT, ENDPOINT)

(4) WE APPROX $\{(x,y) \mid x_i \le x \le x_{i+1},\ y \le f(x)\}$ WITH $\underline{\{(x,y) \mid x_i \le x \le x_{i+1},\ y \le f(x^*_{i,m})\}}$

RECTANGLE!

(5) ADD UP THE AREAS $A_i = f(x^*_{i,m})\,\Delta_x$

(6) LET $m \longrightarrow \infty$

DEFINITION

SUPPOSE THAT FOR ALL CHOICES OF $x_{i,m}^*$ WE HAVE THAT

$$\lim_{m \to \infty} \sum_{i=0}^{m} f(x_{i,m}^*) \underbrace{\frac{b-a}{m}}_{=\Delta x}$$ EXISTS AND IS THE SAME

THEN

$$\int_a^b f(x)\,dx = \lim_{m \to \infty} \sum_{i=0}^{m} f(x_{i,m}^*) \frac{b-a}{m}$$

THIS HAPPENS FOR BASICALLY ALL REASONABLE $f(x)$

THM: SUPPOSE $f(x)$ HAS JUST A FINITE NUMBER OF DISCONTINUITIES ON $[a,b]$ WHICH ARE JUMP DISCONTINUITIES (NO ASYMPTOTES). THEN $\int_a^b f(x)\,dx$ EXISTS

NOTE: $\int_a^b f(x)\,dx$ CAN OFTEN EXIST EVEN IF THE HYPOTHESES ABOVE ARE NOT SATISFIED

WE SAY $f(x)$ IS INTEGRABLE ON $[a,b]$

LEFT ENDPOINTS

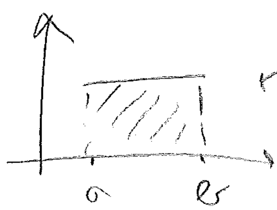
EXAMPLE: $$\int_0^1 x^2+1\,dx = \lim_{m\to\infty} \sum_{i=0}^{m} \frac{1}{m}\left(\frac{i^2}{m^2}+1\right) = \lim_{m\to\infty} 1+\frac{1}{m}\sum_{i=0}^{m} \frac{i^2}{m^2}$$

$$= \lim_{m\to\infty} 1 + \frac{1}{m^3}\left(\frac{1}{6} m(m+1)(2m+1)\right) = 1 + \lim_{m\to\infty} \frac{m\cdot(m+1)(2m+1)}{6m^3} = 1+\frac{1}{3}$$

CAN WE COMPUTE SOME $\int_a^b f(x)\,dx$ ONLY KNOWING THIS? YES!

WE CAN SOLVE THE LIMIT, BUT FOR NOW LET'S USE SOME <u>GEOMETRY</u>:

$$\int_a^b c\,dx \qquad c \text{ CONSTANT}$$



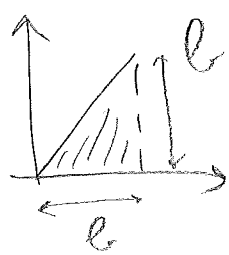
RECTANGLE!

AREA $= (b-a)\cdot c$

IF $c<0$? $(b-a)\cdot c$ AGAIN

$$\int_0^b x\,dx = \frac{b^2}{2}$$

$a=0$

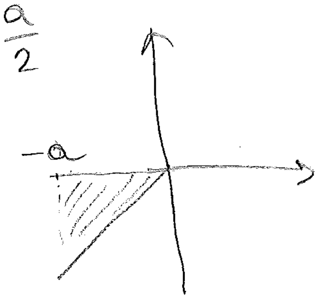


SQUARE TRIANGLE!

AREA $b \cdot \frac{b}{2}$

$$\int_{-a}^0 x\,dx = -\frac{a^2}{2}$$

$b=0$

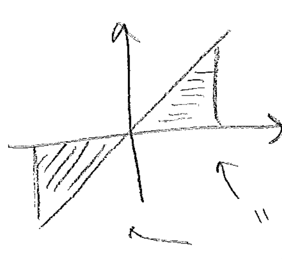


SAME!

AREA $a \cdot \frac{a}{2}$

(EXERCISE: COMPUTE $\int_0^b cx\,dx$)

$$\int_{-a}^a x\,dx = 0$$



"NEGATIVE" AREA AS BIG AS "POSITIVE" AREA

(EXERCISE: COMPUTE $\int_{-a}^a \sin x\,dx$)