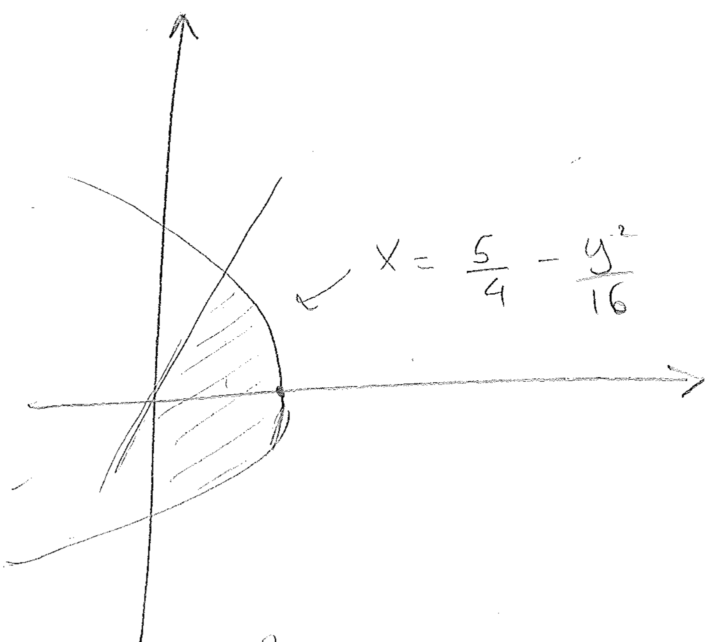
- PRACTICE FINAL: TODAY OR TOMORROW ON WEBSITE. SOL ~ APR 18
- FORMAT OF PRACTICE FINAL AND REAL FINAL ARE THE SAME, DOWN TO NUMBER OF PTS
  - TWO PAGES "ANSWER IN THE BOX"
  - 8 SHORT ANS 3 PTS EA
  - 5 LONG ANS 7 PTS EA
  - WILL COVER ALL MATERIAL, SLIGHT EMPHASIS 11-12. CONTENT UNRELATED TO WHAT IS / IS NOT ON PRACTICE FINAL.
  - 75 PTS
- FRONT PAGE HAS PERSONAL INFO PRE-PRINTED AND VERSION WATERMARK
- SEATS ARE PRE ASSIGNED, IN SRC
- 150 MIN (2 MIN / POINT)
- TERM MARK WILL BE POSTED SOON, UNSCALED
- COURSE PAGE HAS INFO ON SCALING

1) FIND THE VOL OF THE SOLID OBTAINED BY ROTATING THE REGION R CONTAINED BETWEEN THE CURVES

$y^2 = -16x + 20$ $y = 2x$ AND TO THE RIGHT OF THE y-AXIS AROUND THE y-AXIS.



$$\begin{cases} y^2 + 16x - 20 = 0 \\ y = 2x \end{cases}$$

$$\begin{cases} x^2 + 4x - 5 = 0 \\ y = 2x \end{cases}$$

$$\left\{ x = \frac{-4 \pm \sqrt{36}}{2} \right.$$

$$\begin{cases} x = \frac{-5}{1} \\ y = \frac{-10}{2} \end{cases}$$

$$\int_0^2 \left(\frac{5}{4} - \frac{y^2}{16} - \frac{y}{2}\right)^2 \cdot \pi \, dy +$$

$$\int_{-\sqrt{20}}^{0} \left(\frac{5}{4} - \frac{y^2}{16}\right)^2 \cdot \pi \, dy = \left(\frac{23}{20} + \frac{5\sqrt{5}}{3}\right)\pi$$

OTHER OPTION $y = \pm\sqrt{20 - 16x}$

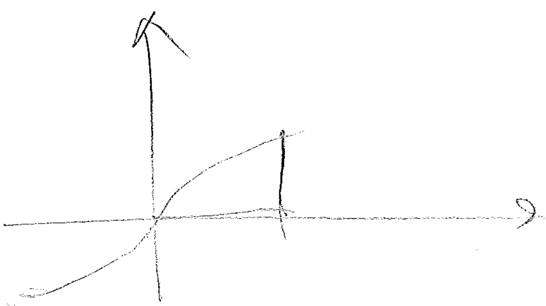
2) FIND THE VOL OF ROTATIONAL SOLID OBTAINED BY ROTATING

$$R = \{0 \leq \operatorname{Tan}(y^2) \leq x \ ; \ 0 \leq x \leq 1\}$$

AROUND THE X-AXIS

$\operatorname{Tan}(y^2) \leq x \sim \quad y^2 \leq \arctan x \sim$

$y \leq \sqrt{\arctan x}$



$$\text{VOL} = \int_0^1 \pi \arctan x \, dx$$

$$\int_0^1 \arctan x \, dx = \Big[\arctan x \cdot x\Big]_0^1 - \int_0^1 \frac{x}{x^2+1} \, dx$$

$$= \frac{\pi}{4} - \int_1^2 \frac{1}{2} \cdot \frac{1}{u} \, du = \frac{\pi}{4} - \frac{\log(2)}{2}$$

SO $\quad \text{VOL} = \dfrac{\pi^2}{4} - \dfrac{\pi \log(2)}{2}$

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

1a. **[2 pts]** Calculate $\int_0^{\pi} x\cos x\,dx$. Simplify your answer completely. Answer: $\boxed{-2}$

$$\int_0^{\pi} x\cos x\,dx = x\sin x\Big|_0^{\pi} - \int_0^{\pi}\sin x\,dx = 0 + \cos(x)\Big|_0^{\pi} = -2$$

$F \quad g$

1b. **[2 pts]** Which integral represents the area between the graphs of $y = 2^{-x}$ and $y = 1 - x$?

Answer: $\boxed{H}$

A: $\int_0^1 \left(2^{-x} - (1-x)\right) dx$ C: $\int_0^1 \left((2^{-x})^2 - (1-x)^2\right) dx$ F: $\int_0^1 \left((1-x) - 2^{-x}\right) dx$

B: $\int_{-1}^0 \left(2^{-x} - (1-x)\right) dx$ D: $\int_{-1}^0 \left((1-x)^2 - (2^{-x})^2\right) dx$ H: $\int_{-1}^0 \left((1-x) - 2^{-x}\right) dx$

1c. **[2 pts]** Which integral equals $\int_0^{\pi/2} f(\sin x)\,dx$? Answer: $\boxed{K}$

J: $\int_0^1 f(u)\,du$ M: $\int_0^{\arcsin(\pi/2)} f(u)\,du$ Q: $\int_0^{\pi/2} f(u)\,du$

K: $\int_0^1 \frac{f(u)}{\sqrt{1-u^2}}\,du$ N: $\int_0^{\arcsin(\pi/2)} \frac{f(u)}{\sqrt{1-u^2}}\,du$ R: $\int_0^{\pi/2} \frac{f(u)}{\sqrt{1-u^2}}\,du$

L: $\int_0^1 f(u)\cos u\,du$ P: $\int_0^{\arcsin(\pi/2)} f(u)\cos u\,du$ S: $\int_0^{\pi/2} f(u)\cos u\,du$

1d. **[2 pts]** Which fraction equals the decimal $0.\overline{321} = 0.321321321\ldots$? Answer: $\boxed{Z}$

T: $1/3$ X: $102/333$

V: $322/999$ Y: $8/27$

W: $321/1000$ Z: $107/333$

$$\sum_{n=1}^{\infty} \frac{321}{1000^n} = \frac{321}{999} = \frac{107}{333}$$

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

2a. **[2 pts]** Only one of the following statements is always true; determine which one is true.

(Assume $f(x)$ and $g(x)$ are continuous functions.) Answer: $\boxed{D}$

A: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. ×

B: If $f(x)$ is an odd function, then $\int_{-3}^{-2} f(x)\,dx = \int_{2}^{3} f(x)\,dx$. ×

C: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. ×

D: If $g(x) \geq f(x) \geq 0$ for all x and $\int_1^\infty g(x)\,dx$ converges, then $\int_1^\infty f(x)\,dx$ converges. ✓

E: $\int_0^7 f(x^2)\,x\,dx = \int_0^7 f(u)\,\frac{du}{2}$. ×

2b. **[2 pts]** Which integral equals $\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right)$? Answer: $\boxed{K}$

F: $\int_0^1 e^{-1-2x}\,dx$

G: $2\int_0^2 e^{-1-2x}\,dx$

H: $\int_0^2 e^{-x}\,dx$

J: $\int_0^2 e^{-1-2x}\,dx$

K: $\int_1^3 e^{-x}\,dx$

2c. **[2 pts]** For which values of the constant q does the sum $\sum_{n=1}^{\infty}\frac{(\log n)^q}{n}$ converge?

L: It converges only when $q < -1$. ✓
M: It converges only when $q \leq -1$.
N: It converges only when $q > -1$.
P: It converges only when $q \geq -1$.
Q: It converges for all q.

Answer: $\boxed{L}$

2d. **[2 pts]** Calculate $\int_{-2}^{2} xe^{x^2}\,dx$. Simplify your answer completely. Answer: $\boxed{0}$

$$\int_{-2}^{2} xe^{x^2} = \frac{1}{2}\int_{-2}^{2}(x^2)'e^{x^2}\,dx = \frac{1}{2}\int_{4}^{4} e^u\,du = 0$$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

3a. **[3 pts]** If $F(x)$ is defined by $F(x) = \int_{x^4-x^3}^{x} e^{\sin t}\,dt$, find $F'(x)$.

$G(x) = \int_0^x e^{\sin T}\,dT \qquad F(x) = G(x) - G(x^4 - x^3)$

$F'(x) = G'(x) - (4x^3 - 3x^2)\,G'(x^4 - x^3) =$

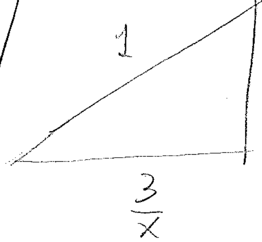
$e^{\sin x} - (4x^3 - 3x^2)\,e^{\sin(x^4 - x^3)}$

3b. **[3 pts]** Evaluate $\int \frac{dx}{x^2\sqrt{x^2-9}}$. Your answer cannot contain any inverse trigonometric functions.

$x = 3\sec(u) \qquad u = \operatorname{arcsec}\left(\frac{x}{3}\right)$

$\int \frac{dx}{x^2\sqrt{x^2-9}} = \frac{1}{9}\int \frac{\tan u \sec u}{(\sec u)^2 \tan u}\,du = \frac{1}{9}\int \frac{1}{(\sec u)}\,du =$

$\left.\frac{\sin u}{9}\right|_{u=\operatorname{arcsec}\left(\frac{x}{3}\right)} = \frac{\sin\left(\operatorname{arcsec}\frac{x}{3}\right)}{9} + C =$



$\sqrt{1 - \frac{9}{x^2}} = \frac{\sqrt{x^2-9}}{x}$

$= \frac{\sqrt{x^2-9}}{9x} + C$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

4a. **[3 pts]** Find the function $y = y(x)$ that satisfies $y(1) = 4$ and

$$\frac{dy}{dx} = \frac{15x^2 + 4x + 3}{y}.$$

$$y' = \frac{15x^2 + 4x + 3}{y} \qquad y \cdot y' = 15x^2 + 4x + 3$$

$$\frac{y^2}{2} = 5x^3 + 2x^2 + 3x + C \qquad y = \pm\sqrt{10x^3 + 4x^2 + 6x + C}$$

$$y(1) = \sqrt{10 + 4 + 6 + C} = \sqrt{20 + C} \quad \text{So } C = -4$$

4b. **[3 pts]** Calculate $\displaystyle\int \frac{12x + 4}{(x-3)(x^2+1)}\, dx.$

$$\frac{12x + 4}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{B + Cx}{x^2+1}$$

$$A = \frac{12 \cdot 3 + 4}{3^2 + 1} = \frac{40}{10} = 4$$

$$12x + 4 = 4(x^2+1) + (x-3)(B + Cx) = (4 + C)x^2 + (B - 3C)x +$$

$$C = -4 \qquad B = 0$$

$$\int \frac{4}{x-3} + \frac{-4x}{x^2+1}\, dx = 4\log|x-3| - 2\log(x^2+1) + C$$

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

5a. **[3 pts]** Does $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converge or diverge? Justify your conclusion.

INTEGRAL TEST $\quad u = \sqrt{x}$

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = -\int_1^\infty e^{-u}\,du$$

$$= \lim_{R\to\infty} e^{-1} - e^{-R} = e^{-1}$$ SO IT CONVERGES

$\frac{e^{-\sqrt{x}}}{\sqrt{x}}$ IS DECR AS $e^{-\sqrt{x}}$ IS DEC AND $\sqrt{x}$ IS INCR.

5b. **[3 pts]** Does $\sum_{n=1}^{\infty} \frac{n^4 2^{n/3}}{(2n+7)^4}$ converge or diverge? Justify your conclusion.

RATIO TEST:
$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^4\, 2^{\frac{n+1}{3}}}{(2n+2+7)^4}}{\frac{n^4\, 2^{n/3}}{(2n+7)^4}} \right| = \lim_{n\to\infty} 2^{\frac{1}{3}} \left| \frac{\frac{(n+1)^4}{(2n+2+7)^4}}{\frac{n^4}{(2n+7)^4}} \right|$$

$$= 2^{\frac{1}{3}} > 1$$ SO IT DIVERGES

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

6a. **[3 pts]** Find the sum of the convergent series $\sum_{n=3}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)$. Simplify your answer completely.

$S_N = \cos\frac{\pi}{3} - \cos\frac{\pi}{4} + \cos\frac{\pi}{4} - \cos\frac{\pi}{5} + \cos\frac{\pi}{5} - \ldots - \cos\frac{\pi}{N+1} =$

$\cos\frac{\pi}{3} - \cos\frac{\pi}{N+1}$ so $\lim_{N\to\infty} S_N = \cos\frac{\pi}{3} - \cos(0)$

$= \frac{1}{2} - 1 = -\frac{1}{2}$

6b. **[3 pts]** Determine the interval of convergence for the power series $\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}}$.

$R=1$, $c=1$ AT 0 CONV BY ALT TEST, AT 2 DIV BY P-TEST

SO INTERVAL IS $[0, 2)$

Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

7. **[7 pts]** Find the average value of the function $f(x) = 3\cos^3 x + 2\cos^2 x$ on the interval $0 \le x \le \frac{\pi}{2}$. Simplify your answer completely.

$$3\cos^3(x) + 2\cos^2(x) = \cos(2x) + 1 + 3(1 - \sin^2(x))\cos x$$

$$= \cos(2x) + 1 + 3\cos(x) - 3\sin^2(x)\cos(x) =$$

$$\int \cos(2x) + 1 + 3\cos(x) - 3\sin^2(x)\cos x \, dx =$$

$$\frac{\sin 2x}{2} + x + 3\sin x - \int 3u^2\, du =$$

$$\frac{\sin 2x}{2} + x + 3\sin x - \sin^3(x)$$

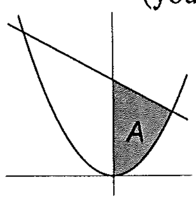
So $$\text{AVG} = \frac{2}{\pi}\left(\frac{\sin 2x}{2} + x + 3\sin x - \sin^3(x)\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$\frac{2}{\pi}\left(0 + \frac{\pi}{2} + 3 - 1\right) - \frac{2}{\pi}\left(0 + 0 + 0 - 0\right) =$$

$$1 + \frac{4}{\pi}$$

8. Let A be the region to the right of the y-axis that is bounded by the graphs of $y = x^2$ and $y = 6 - x$. Both parts of this question concern this region A.

(a) **[4 pts]** Find the centroid of A, assuming it has constant density $\rho = 1$. The area of A is $\frac{22}{3}$ (you don't have to show this). You may leave your answer in calculator-ready form.



$$\begin{cases} y = x^2 \\ y = 6-x \end{cases} \sim \begin{cases} 6-x = x^2 \end{cases} \sim x^2+x-6=0 \sim x = \frac{-1\pm\sqrt{25}}{2} = \begin{matrix} 2 \\ -3 \end{matrix}$$

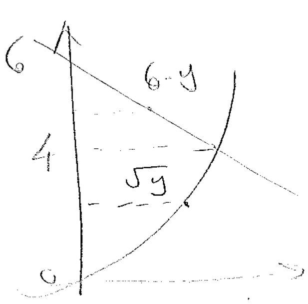
$$\bar{x} = \frac{1}{A}\int_0^2 x(6-x-x^2)\,dx = \frac{1}{A}\left(3x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^2 =$$

$$= \frac{3}{22}\left(12 - \frac{8}{3} - 4\right) = \frac{3}{22} \cdot \frac{16}{3} = \frac{8}{11}$$

$$\bar{y} = \frac{1}{2A}\int_0^2 (6-x)^2 - x^4\,dx = \frac{1}{2A}\left(36x - 6x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^2$$

$$= \frac{3}{44}\left(72 - 24 + \frac{8}{3} - \frac{16}{5}\right) = \frac{664}{220}$$

(b) **[3 pts]** Write down an expression, using horizontal slices (disks), for the volume obtained when the region A is rotated around the y-axis. Do not evaluate any integrals; simply write down an expression for the volume.



$$\int_0^4 \pi\cdot(\sqrt{y})^2\,dy + \int_4^6 \pi(6-y)^2\,dy$$

$$= \pi\int_0^4 y\,dy + \pi\int_4^6 y^2 - 12y + 36\,dy =$$

$$8\pi + \frac{8}{3}\pi = \frac{32}{3}\pi$$

9. Both parts of this question concern the integral $I = \int_0^2 (x-3)^5\,dx$.

(a) **[3 pts]** Write down the Simpson's Rule approximation to I with $n = 6$. You may leave your answer in calculator-ready form.

$$x_0, \ldots, x_6 = 0, \tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{4}{3}, \tfrac{5}{3}, 2 \qquad \Delta x = \tfrac{1}{3}$$

$$S_6 = \frac{1}{9}\left(-3^5 + 4\left(-\tfrac{8}{3}\right)^5 + 2\left(-\tfrac{7}{3}\right)^5 + 4(-2)^5 + 2\left(-\tfrac{5}{3}\right)^5 + 4\left(-\tfrac{4}{3}\right)^5 - 1\right)$$

(b) **[4 pts]** Which method of approximating I results in a smaller error bound: the Midpoint Rule with $n = 100$ intervals, or Simpson's Rule with $n = 10$ intervals? Justify your answer. You may use the formulas

$$|E_M| \le \frac{K(b-a)^3}{24n^2} \qquad \text{and} \qquad |E_S| \le \frac{L(b-a)^5}{180n^4},$$

where K is an upper bound for $|f''(x)|$ and L is an upper bound for $|f^{(4)}(x)|$.

$$K = \max_{[0,2]} |20(x-3)^3| = \max_{[0,2]} \underset{\uparrow \text{ DECR}}{-20(x-3)^3}$$

$$= 20 \cdot 27 \qquad |E_M| \le \frac{20 \cdot 27 \cdot (2-0)^3}{24 \cdot 100^2} = \frac{9}{500}$$

$$L = \max_{[0,2]} |120(x-3)| = \max_{[0,2]} \underset{\uparrow \text{ DECR}}{-120(x-3)}$$

$$= 360$$

$$|E_S| \le \frac{360}{180} \cdot \frac{2^5}{10^4} = \frac{2^6}{10^4} = \frac{4}{5^4} = \frac{4}{625}$$

$|E_S| < |E_M|$ so SIMPSON n=10 BETTER

10. Define $f(x) = \int_0^x \frac{1-e^{-t}}{t}\, dt$.

(a) **[4 pts]** Show that the Maclaurin series for $f(x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$.

$$e^{-T} = \sum_{m=0}^{\infty} \frac{(-1)^m T^m}{m!} \qquad \int_0^x \frac{1-e^{-T}}{T}\, dT = \int_0^x \sum_{m=1}^{\infty} \frac{(-1)^{m-1} T^{m-1}}{m!}\, dT$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1} T^m}{m! \cdot m} \Bigg|_0^x = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m! \cdot m}$$

(b) **[3 pts]** Use the Ratio Test to answer the question: for which values of x does the Maclaurin series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$ converge?

$$\lim_{m \to \infty} \left| \frac{(-1)^m}{(m+1)(m+1!)} \cdot \frac{m \cdot m!}{(-1)^{m-1}} \right| = \lim_{m \to \infty} \frac{m}{(m+1)^2} = 0$$

$R = \infty$

11. Both parts of this question concern the series $S = \sum_{n=1}^{\infty} (-1)^{n-1} 24n^2 e^{-n^3}$.

(a) **[4 pts]** Show that the series S converges absolutely.

RATIO TEST $\lim_{n \to \infty} \left| \frac{24(n+1)^2 e^{-(n+1)^3}}{24 n^2 e^{(n^3)}} \right| =$

$\lim_{n \to \infty} \left| \frac{(n+1)^2}{n^2} \right| \left| \frac{e^{-(n+1)^3}}{e^{n^3}} \right| = \lim_{n \to \infty} \frac{e^{-(n^3+3n^2+3n+1)}}{e^{n^3}}$

$\uparrow$

1 $= \lim_{n \to \infty} e^{-3n^2-3n-1} = 0$

(b) **[3 pts]** Suppose that you approximate the series S by its fifth partial sum S_5. Give an upper bound for the error resulting from this approximation. Explain why your error bound is valid for this series.

SERIES IS ALTERNATING + DECREASING

SO $|S - S_N| \leq |a_{N+1}|$ $|S - S_5| \leq |a_6| =$

$24(6)^2 \cdot e^{-6^3}$ BY ALT SERIES

THM.

DECREASING: $(x^2 e^{-x^3})' = 2x e^{-x^3} - 3x^3 e^{-x^3} < 0$

FOR $x \geq 1$