WARM UP:

i) FIND A POWER SERIES $f(x) = \sum_{m=0}^{\infty} A_m x^m$ SUCH THAT $f(x)$ CONVERGES ON $[-1, 1]$ BUT $\frac{d}{dx} f(x)$ DOES NOT CONVERGE AT EITHER -1 OR 1

ii) FIND A POWER SERIES $f(x) = \sum_{m=0}^{\infty} A_m x^m$ SUCH THAT $\int f(x)\, dx$ CONVERGES ON $[-1, 1)$ (OR $(-1, 1]$) BUT $f(x)$ CONVERGES ON $(-1, 1)$ ONLY

SOL:

i) WE WANT $R = 1$, SO WE CAN PICK A_m TO BE SOME POLYNOMIAL FUNCTION OF m. TAKING $\frac{d}{dx}$ MULTIPLIES THE mTH COEFF. BY m, SO WE CAN JUST TAKE A_m SUCH THAT $\sum A_m$ CONV. ABS. BUT $\sum m A_m$ DOESN'T, E.G. $A_m = \frac{1}{m^2}$

SO $f(x) = \sum_{m=0}^{\infty} \frac{x^m}{m^2}$ CONV ON $[-1, 1]$,

$\frac{d}{dx} f(x) = \sum_{m=0}^{\infty} \frac{(m+1) x^m}{m^2}$ DIVERGES AT $x = 1$.

ii) INTEGRATING DIVIDES THE COEFFICIENTS BY m, SO WE CAN TAKE $A_m = 1$ FOR ALL m. THEN

$$\int f(x)\,dx = \sum_{m=1}^{\infty} \frac{x^m}{m}$$ CONVERGES AT

$X = -1$ BY THE ALTERNATING TEST, WHILE

$$f(x) = \sum_{m=1}^{\infty} x^m$$ DIVERGES AT ± 1 BY

THE DIVERGENCE TEST.

SO HOW DO WE GET TO $\frac{1}{(1-x)^2}$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) \quad !$$

SO $$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \sum_{n=0}^{\infty}(n+1)x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots \quad \underline{\text{ON } (-1,1)} \quad (\text{CHECK!})$$

<u>EXAMPLE</u> $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

HOW ABOUT e^x?

LET'S LOOK AT $\sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

WE HAVE $$\frac{d}{dx} f(x) = \sum_{n=0}^{\infty} \frac{n+1}{(n+1)!} \cdot x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad !!$$

SO $\frac{d}{dx} f(x) = f(x)$, AND $f(0) = 1$... THEN

$f(x) = e^x$!! <u>EVERYWHERE.</u>

<u>THEOREM</u> (SUBSTITUTION IN POWER SERIES)

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ HAS RADIUS OF CONV. R, THEN

- $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$ HAS R.O.C. $\frac{R}{k}$
- $f(x^m) = \sum_{n=0}^{\infty} a_n x^{m \cdot n}$ HAS R.O.C. $\sqrt[m]{R}$

$$= \sum_{k=0}^{\infty} \left(\begin{cases} 0 & \text{IF } k \neq m \cdot n \\ a_n & \text{IF } k = m \cdot n \end{cases} \right) \cdot x^k$$

NOTE: YOU CAN CHANGE THE CENTER OF A POW. SER. BY SUBSTITUTING $X \to X+a$, FOR EXAMPLE IF $f(x) = \sum_{m=0}^{\infty} A_m (x-c)^m$ THEN $f(x+c) = \sum_{m=0}^{\infty} A_m x^m$ HAS CENTER 0.

EXAMPLE: $\frac{x^2}{3+x}$ AS A POW. SER.

$$\frac{x^2}{3+x} = x^2 \cdot \left(\frac{1}{3+x}\right) = \frac{x^2}{3} \cdot \frac{1}{1+\frac{x}{3}}$$

BUT $\frac{1}{1+\frac{x}{3}} = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{3}\right)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{3^m} x^m$

SO $\frac{x^2}{3+x} = \frac{x^2}{3} \cdot \frac{1}{1+\frac{x}{3}} = \frac{x^2}{3} \sum_{m=0}^{\infty} \frac{(-1)^m}{3^m} x^m =$

$$3 \cdot \left(\frac{x}{3}\right)^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{3^m} x^m = \sum_{m=2}^{\infty} \frac{(-1)^m}{3^{m-1}} x^m \text{ ON } \underline{(-3, 3)}$$

EXAMPLE $\frac{1}{1+x^2}$ AS A POW. SER.

WE KNOW $f(x) = \frac{1}{1+x} = \sum_{m=0} (-1)^m x^m$

SO $f(x^2) = \frac{1}{1+x^2} = \sum_{m=0}^{\infty} (-1)^m x^{2m}$ ON $\underline{(-1, 1)}$

Q: WHY? $\frac{1}{1+x^2}$ IS WELL DEFINED FOR ALL X... THINK ABOUT IT!

EXAMPLE: arctan(x) AS A POW. SER.

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \sum_{m=0}^{\infty} (-1)^m x^{2m} \quad \text{SO}$$

$$C + \arctan(x) = \int \frac{1}{1+x^2}\,dx = C + \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{2m+1} x^{2m+1}$$

$$\text{SO} \quad \arctan(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{2m+1} x^{2m+1} \quad \underline{\text{ON}}$$

$\underline{[-1, 1]}$ (CHECK!)

EXAMPLE: WHAT IS THE VALUE OF

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2m+1} \,? \qquad \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2m+1} = \arctan(1) = \frac{\pi}{4}$$

EXAMPLE: WHAT IS THE VALUE OF $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m 2^m}{(m-1)!}$

IDEA: IT LOOKS LIKE $\sum \frac{1}{m!} \ldots$

$$\sum_{m=1}^{\infty} \frac{(-1)^m 2^m}{(m-1)!} = \sum_{m=1}^{\infty} \frac{(-2)^m}{(m-1)!} = \frac{-2}{0!} + \frac{(-2)^2}{1!} + \frac{(-2)^3}{2!} + \frac{(-2)^4}{3!} + \ldots$$

$$= -2 \cdot \sum_{m=0}^{\infty} \frac{(-2)^m}{m!} = -2e^{-2}$$

TAYLOR SERIES

TWO QUESTIONS:

i) IF WE KNOW THAT $f(x) = \sum_{m=0}^{\infty} A_m x^m$

AROUND $x = 0$, CAN WE FIND $\frac{d^m}{dx^m} f(0)$?

ii) IF WE KNOW $f(0)$ AND $\frac{d^m}{dx^m} f(0)$ FOR ALL m, CAN WE FIND THE COEFFICIENTS A_m?

WELL,

i) IF $f(x) = A_0 + A_1 x + A_2 x^2 + \ldots$ THEN

$\frac{d}{dx} f(x) = A_1 + 2A_2 x + 3A_3 x^3 + \ldots$ AND

$\frac{d^m}{dx} f(x) = 2 \cdot 3 \cdot \ldots \cdot m A_m + 2 \cdot 3 \cdot \ldots \cdot m{+}1 A_{m+1} x + \ldots$

SO $f(0) = 2 \cdot 3 \cdot \ldots \cdot m A_m + 0 + 0 + \ldots$

$= 2 \cdot 3 \cdot \ldots m A_m = m! A_m$

ii) CONVERSELY $A_m = \frac{d^m}{dx} f(0) \cdot \frac{1}{m!}$

THIS CLEARLY WORKS FOR ANY CENTER c, GIVING US

$$A_m = \frac{d^m}{dx} f(c) \cdot \frac{1}{m!}$$

SO THAT AROUND c WE HAVE

$$f(x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n} f(c)\,(x-c)^n =$$

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3$$

$$+ \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

WAIT, THIS LOOKS FAMILIAR!

RECALL:

THE TAYLOR POLYNOMIAL OF DEGREE n CENTERED AT c FOR $f(x)$ IS

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c) + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

IT APPROXIMATES $f(x)$ WITH AN ERROR

$$|E_n(x)| = |f(x) - T(x)| \leq \left| \frac{M}{(n+1)!}(x-c)^{n+1} \right|$$

WHERE $M = \mathrm{MAX}_{[c,x]} f^{(n+1)}$

WE CAN NOW EXTEND THIS POWERFUL IDEA!

THEOREM: (TAYLOR SERIES)

THE TAYLOR SERIES (EXPANSION) FOR $f(x)$

WITH CENTER c IS

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

$$= \sum_{n=0}^{\infty} \frac{d^n}{dx^n} f(c) \cdot \frac{(x-c)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!}$$

- FOR ALL THE FUNCTIONS $f(x)$ WE'LL BE CONCERNED ABOUT WE HAVE

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(x-c)^n$$

WHEREVER THE SERIES CONVERGES

- MOREOVER, IF WE KNOW ALREADY THAT

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n$$

THEN THIS IS ITS TAYLOR SERIES AND

$$A_n = \frac{f^{(n)}(c)}{n!}$$

EXAMPLE: WE HAVE ALREADY DERIVED THE FOLLOWING EQUALITIES:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} x^{2m-1}$$

BY THE SECOND PART OF THE THM

THESE ARE THE TAYLOR EXPANSIONS OF

$\frac{1}{1-x}$, e^x, $\log(1+x)$, $\arctan(x)$ AT $c=0$

NOTE THAT ALSO (BY $x \to x-1$)

$$\log(x) = x-1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

EXAMPLE: EXPANSION OF $\sin(x)$ AT $c=0$

WE HAVE $(\sin(x))' = \cos(x)$, $(\cos(x))' = -\sin(x)$

SO THE SEQUENCE OF $f^{(m)}(c)$ AT $c=0$

GOES $0, 1, 0, -1, 0, 1, 0, -1$ SO THE TAYLOR

SERIES IS

$$0 + x + 0 - \frac{x^3}{6} + 0 + \frac{x^5}{120} + 0 - \dots = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)!} x^{2m-1}$$

THIS CONVERGES <u>EVERYWHERE</u>. WHY?

IDEA 1: COMPARE WITH $\sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$

IDEA 2: $$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^{2m-1}}{(2m-1)!} = x \sum_{m=0}^{\infty} \frac{(-1)^m (x^2)^m}{(2m+1)!}$$

NOW BY RATIO TEST (ON COEFFICIENTS) $\sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(2m+1)!}$ HAS

$R = \infty$, AND WE CAN APPLY THE SUBSTITUTION THM

IDEA 3: DIRECTLY APPLY RATIO TEST ON WHOLE

$$\sum_{m=0}^{\infty} \frac{(-1)\, x^{2m-1}}{(2m-1)!} \quad ! \quad \lim_{m \to \infty} \left| \frac{\frac{x^{2m+1}}{2m+1!}}{\frac{x^{2m-1}}{2m-1!}} \right| = \lim_{m \to \infty} \frac{x^2}{(2m+1)(2m)}$$

$= 0$

EXAMPLE: EXPANSION OF $\cos(x)$ AT $c=0$

THE VALUES $f^{(m)}(0)$ GO $1, 0, -1, 0, 1, 0, -1, \ldots$

SO $$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2m!}$$

EXERCISE: USE THE IDEAS ABOVE TO SHOW THAT THIS POW. SER. HAS $R = \infty$:

EXAMPLE: EXPANSION OF e^{x^3} AT $c=0$

DO WE HAVE TO TAKE A LOT OF UGLY DERIVATIVES?

NO! WE HAVE $$e^{x^3} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \ldots$$

$= \sum_{m=0}^{\infty} \frac{x^{3m}}{m!}$ BY SUBSTITUTION, AND BY THM THIS IS THE TAYLOR SERIES!

EXAMPLE: FIND $\frac{d^9}{dx^9} e^{x^3}(0)$

WE KNOW THAT $e^{x^3} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \dots$

$= 1 + 0 + 0 + x^3 + 0 + 0 + \frac{x^6}{2} + 0 + 0 + \frac{x^9}{6}$

SO IN THE TAYLOR EXPANSION $A_9 = \frac{1}{6}$

BUT $A_9 = \frac{d^9}{dx^9} e^{x^3}(0) \cdot \frac{1}{9!}$ SO

$$\frac{1}{6} = \frac{(e^{x^3})^{(9)}(0)}{9!} \sim (e^{x^3})^{(9)}(0) = \frac{9!}{6}$$

EXAMPLE: FIND THE COEFFICIENT OF x^4 IN THE TAYLOR SERIES OF

$$\frac{e^x - \cos(x) - \sin(x)}{x^2} \quad \text{AT} \quad c = 0$$

LET'S TRY TO REWRITE THIS A SERIES DIVIDED BY x^2

$$\frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right) - \left(x - \frac{x^3}{6} + \dots\right)}{x^2}$$

$$= \frac{0 + 0 + x^2 + \frac{x^3}{3} + 0 + 0 + \frac{x^6}{360} + \dots}{x^2} = 1 + \frac{x}{3} + \frac{x^4}{360} + \dots$$

SO $A_4 = \frac{1}{360}$

EXAMPLE: SOLVE $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$ WITHOUT USING L'HOPITAL.

WE CAN REWRITE $\frac{\cos(x)-1}{x^2} = \frac{\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\dots\right)-1}{x^2}$

$$= \frac{-\frac{x^2}{2}+\frac{x^4}{24}-\dots}{x^2}$$

SO $\lim_{x\to 0} \frac{\cos(x)-1}{x^2} = \lim_{x\to 0} \frac{-\frac{x^2}{2}+\frac{x^4}{24}-\dots}{x^2}$

$$= \lim_{x\to 0} -\frac{1}{2} + \underbrace{x^2}_{\to 0}\underbrace{\left(\frac{1}{24}-\dots\right)}_{\text{CONV. TO SOME VALUE}} = -\frac{1}{2}$$